

RATIONAL CHEREDNIK ALGEBRAS AND SCHUBERT CELLS

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ABSTRACT. In this article we explore the relationship between the representation theory of the rational Cherednik algebras of type A at $t = 0$ and the geometry of the Calogero-Moser space. We give a representation theoretic interpretation of fact that the Calogero-Moser space can be described as a certain (infinite) union of Schubert cells. We describe the Tor and Ext groups of generalized Verma modules and show that they are Gesteinhaber algebras, resp. Gesteinhaber modules.

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1. INTRODUCTION

One of the many things that rational Cherednik algebras are related to is the Calogero-Moser integrable system. Namely, it was shown in the original paper [12], where rational Cherednik algebras were first defined by Etingof and Ginzburg, that the centre of the rational Cherednik algebra of type A , at $t = 0$, is isomorphic to the coordinate ring of Wilson's completion of the Calogero-Moser phase space. The Calogero-Moser space is also closely related to the adelic Grassmannian and rational solutions of the KdV hierarchy. As such, natural objects of study of the KdV hierarchy such as the τ and Baker functions and Schubert cells appear naturally in the setting of the Calogero-Moser space. The purpose of this article is to try and understand how these objects manifest themselves in terms of the representation theory of rational Cherednik algebras.

In the remainder of the introduction, we outline the main results of the paper, introducing the notation as it is needed.

1.1. The rational Cherednik algebra. We begin by studying, independent of any connection to the Calogero-Moser space, certain naturally defined representation of the rational Cherednik algebra. Let H_n be the rational Cherednik algebra associated to the symmetric group \mathfrak{S}_n at $t = 0$ and $c = -2$, see (2.1) for precise definitions. The main objects of study in this paper will be certain generalized Verma modules for H_n . Let \mathfrak{h}^* denote the dual of the permutation representation for \mathfrak{S}_n . For each $p \in \mathfrak{h}^*$, the skew-group ring $\mathbb{C}[\mathfrak{h}^*] \rtimes \mathfrak{S}_p$, where \mathfrak{S}_p is the stabilizer of p in \mathfrak{S}_n , is a subalgebra of H_n . If $b = \sum_{i=1}^k n_i p_i$ is the image of p in $\mathfrak{h}^*/\mathfrak{S}_n$, then the irreducible \mathfrak{S}_p -modules are naturally labeled by multi-partitions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(k)})$, where $\lambda^{(i)}$ is a partition of n_i . We define the generalized Verma module for H_n , labeled by p and λ , to be

$$\Delta(p, \lambda) := H_n \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes \mathfrak{S}_p} \lambda, \quad (1.1.1)$$

where elements in $\mathbb{C}[\mathfrak{h}^*]$ act on λ by evaluation at p . The central result of this article is an explicit description of the endomorphism ring $\text{End}_{H_n}(\Delta(p, \lambda))$ of the module $\Delta(p, \lambda)$. The following is a special case of Theorem 3.3.9.

Theorem 1.1.2. *For all $p \in \mathfrak{h}^*$ and $\lambda \in \text{Irr}(\mathfrak{S}_p)$,*

- (1) *The centre Z_n of H_n surjects onto $E_{(p, \lambda)} := \text{End}_{H_n}(\Delta(p, \lambda))$.*
- (2) *The algebra $E_{(p, \lambda)}$ is a polynomial ring of dimension n .*
- (3) *$\Delta(p, \lambda)$ is a cyclic $E_{(p, \lambda)}$ -module.*

Let $H_\lambda(q)$ be the hook polynomial associated to the partition λ . The algebra H_n is \mathbb{Z} -graded. The Verma module $\Delta(\lambda) := \Delta(0, \lambda)$ is a graded H_n -module. This implies that E_λ is also \mathbb{Z} -graded. Theorem 3.1.3 implies that

Corollary 1.1.3. *The algebra E_λ is \mathbb{N} -graded with character*

$$\text{ch}_q(E_\lambda) = \frac{1}{H_\lambda(q)}.$$

In fact, we can do much better than Theorem 1.1.2, and consider not only $\text{End}_{H_n}(\Delta(p, \lambda))$, but the whole ext algebra $\text{Ext}_{H_n}^\bullet(\Delta(p, \lambda))$. Similarly, if we define $\Delta(p, \lambda)^{op}$ to be the right H_n -module induced as in (1.1.1) from the right $\mathbb{C}[\mathfrak{h}^*] \rtimes \mathfrak{S}_p$ -module λ , then we can also consider the tor algebra $\text{Tor}_{H_n}^\bullet(\Delta(p, \lambda)^{op}, \Delta(p, \lambda))$. By Theorem 1.1.2, the algebra $E_{(p, \lambda)}$ can be identified with Z_n/I where $I = \text{Ann}_{Z_n} \Delta(p, \lambda)$. Since I only depends on b and λ , we set $\Omega_{b, \lambda} := \text{Spec } E_{(p, \lambda)}$, which is a smooth, closed subvariety of $X_n := \text{Spec } Z_n$. Then, $N_{b, \lambda}^\vee := I/I^2$ is a locally free $E_{(b, \lambda)}$ -module. It is the module of sections of the conormal bundle of $\Omega_{b, \lambda}$ in X_n . Its dual $N_{b, \lambda} := (I/I^2)^\vee$ is the module of sections of the normal bundle of $\Omega_{b, \lambda}$ in X_n . The following theorem is an application of the theory developed in [1].

Theorem 1.1.4. *The algebra $\text{Tor}_{H_n}^\bullet(\Delta(p, \lambda), \Delta(p, \lambda))$ admits a canonical structure of a Gesteinhaber algebra such that*

$$\text{Tor}_{H_n}^\bullet(\Delta(p, \lambda)^{op}, \Delta(p, \lambda)) \simeq \wedge^\bullet N_{p, \lambda}^\vee \quad (1.1.5)$$

as Gesteinhaber algebras.

Moreover, the space $\text{Ext}_{H_n}^\bullet(\Delta(p, \lambda), \Delta(p, \lambda))$ admits a canonical structure of Gesteinhaber module over the Gesteinhaber algebra $\text{Tor}_{H_n}^\bullet(\Delta(p, \lambda)^{op}, \Delta(p, \lambda))$ such that

$$\text{Ext}_{H_n}^\bullet(\Delta(p, \lambda), \Delta(p, \lambda)) \simeq \wedge^\bullet N_{p, \lambda}$$

as Gesteinhaber modules, compatible in the obvious sense with the identification (1.1.5).

Theorem 1.1.2, Corollary 1.1.3 and Theorem 1.1.4 are valid for any rational Cherednik algebra, provided the support of the module $\Delta(p, \lambda)$ is contained in the smooth locus of the generalized Calogero-Moser space.

1.2. The Calogero-Moser space. Wilson's completion of the Calogero-Moser space can be described as follows, see section 4 for details. Let $\overline{\text{CM}}_n$ be the set of all pairs of $n \times n$, complex matrices (X, Y) such that the rank of $[X, Y] + I_n$ is one. The group $\text{PGL}_n(\mathbb{C})$ acts on the space $\overline{\text{CM}}_n$ and the Calogero-Moser space CM_n is defined to be the categorical quotient $\overline{\text{CM}}_n / \text{PGL}_n$. It is a smooth, $2n$ -dimensional affine variety. As noted above, Etingof and Ginzburg constructed an isomorphism $X_n \xrightarrow{\sim} \text{CM}_n$. On the other hand, Wilson showed that the union over all n of the Calogero-Moser spaces CM_n can be identified with a certain infinite dimensional space, the *adelic Grassmannian*. Thus, there is an embedding of the space X_n into this adelic Grassmannian. In order to be able to describe the image of the subspaces $\Omega_{b, \lambda}$ in the adelic Grassmannian, it is more convenient to describe the adelic Grassmannian in terms of certain spaces of quasi-exponentials.

A holomorphic function f on the complex plane that can be expressed as

$$f(x) = e^{b_1 x} g_1(x) + \cdots + e^{b_k x} g_k(x),$$

where $b_i \in \mathbb{C}$ and $g_i(x)$ is a polynomial, is called a quasi-exponential function. Let \mathcal{Q} denote the space of all quasi-exponential functions. A finite dimensional subspace C of \mathcal{Q} is called *homogeneous* if $C = \bigoplus_{b \in \mathbb{C}} C_b$, where C_b is spanned by functions of the form $e^{bx} g(x)$ for some polynomial $g(x)$. The set of all finite dimensional, homogeneous subspaces of \mathcal{Q} is denoted \mathcal{QGr} . Using the Wronskian, one can pick out certain distinguished spaces in \mathcal{QGr} called *canonical* spaces. The set of all canonical spaces is denoted \mathcal{QE} . Wilson showed [33] that each point in the Calogero-Moser space CM_n can be labeled by a canonical space $C \in \mathcal{QE}$. As a consequence we have bijections

$$\begin{array}{ccccc} X_n & \xrightarrow{\sim} & \text{CM}_n & \xrightarrow{\sim} & \mathcal{QE}_n \\ & & \searrow \nu_n & & \end{array}$$

where \mathcal{QE}_n is the set of all n -dimensional spaces in \mathcal{QE} . One of the main goals of the paper is to describe the image of $\Omega_{b,\lambda}$ under ν_n .

For $\mathbf{b} = \sum_{i=1}^k n_i b_i \in \mathfrak{h}^*/\mathfrak{S}_n$, we define

$$\text{Gr}_{\mathbf{b}}(\mathcal{QGr}) = \text{Gr}_{n_1}(e^{b_1 x} \mathbb{C}[x]_{2n_1}) \times \cdots \times \text{Gr}_{n_k}(e^{b_k x} \mathbb{C}[x]_{2n_k}),$$

a projective subvariety of \mathcal{QGr} , where $\mathbb{C}[x]_k$ is the space of all polynomials of degree less than k . Each $\text{Gr}_{n_i}(e^{b_i x} \mathbb{C}[x]_{2n_i})$ has a natural stratification by Schubert cells, which are labeled by all those partitions μ that fit into a square of length $2n_i$. Thus, if $\lambda = (\lambda^{(1)}, \dots, \lambda^{(k)})$ with $\lambda^{(i)} \vdash n_i$, then

$$\Omega_{\mathbf{b},\lambda}^{\text{qe}} = \Omega_{b_1,\lambda^{(1)}}^{\text{qe}} \times \cdots \times \Omega_{b_k,\lambda^{(k)}}^{\text{qe}}$$

is a locally closed subvariety of $\text{Gr}_{\mathbf{b}}(\mathcal{QGr})$.

Theorem 1.2.1. *Let $p \in \mathfrak{h}^*$ and $\lambda = (\lambda^{(1)}, \dots, \lambda^{(k)})$ an irreducible \mathfrak{S}_p -module. Then, the map ν_n restricts to an isomorphism of varieties*

$$\nu_n : \Omega_{\mathbf{b},\lambda} \xrightarrow{\sim} \Omega_{\mathbf{b},\lambda^t}^{\text{qe}},$$

where \mathbf{b} is the image of p in $\mathfrak{h}^*/\mathfrak{S}_n$, and λ^t denotes componentwise transpose.

The proof of Theorem 1.2.1 is given in section 7.1. The essential fact that we shall repeatedly use in the proof of Theorem 1.2.1 is that each of the spaces X_n , CM_n and \mathcal{QE}_n satisfies a certain factorization property. Namely, there is a map from each of the spaces to $\mathfrak{h}^*/\mathfrak{S}_n$,

$$\begin{array}{ccccc} X_n & \xrightarrow{\sim} & \text{CM}_n & \xrightarrow{\sim} & \mathcal{QE}_n \\ & \searrow \pi & \downarrow \pi^{\text{cm}} & \swarrow \text{Supp} & \\ & & \mathfrak{h}^*/\mathfrak{S}_n & & \end{array}$$

such that the fiber of each map over \mathbf{b} can be factorized as the product of the fibers over $n_i \cdot b_i$, where i runs over $1, \dots, k$, for instance

$$\pi^{-1}(\mathbf{b}) \simeq \pi^{-1}(n_1 \cdot b_1) \times \cdots \times \pi^{-1}(n_k \cdot b_k),$$

where $\pi^{-1}(n_i \cdot b_i)$ is a closed subvariety of X_{n_i} . The key step in our work is to show that each of the isomorphisms $X_n \xrightarrow{\sim} \text{CM}_n$ and $\text{CM}_n \xrightarrow{\sim} \mathcal{QE}_n$ is compatible with these factorizations, in the obvious sense. A closely related result [29] appeared whilst this paper was in preparation. The second key fact that we shall repeatedly use is that each of the spaces X_n , CM_n and \mathcal{QE}_n is equipped with a canonical \mathbb{C}^\times -action such that the isomorphisms between them is \mathbb{C}^\times -equivariant.

1.3. Dual to the space $\Omega_{b,\lambda}$ is a space $\mathcal{U}_{a,\mu}$, where $a = \sum_{i=1}^l n_i a_i \in \mathfrak{h}/\mathfrak{S}_n$. It is the support of a dual Verma module, $\nabla(q, \mu)$, where $q \in \mathfrak{h}$ with $\bar{q} = a$ and $\mu = (\mu^{(1)}, \dots, \mu^{(l)})$ is an irreducible \mathfrak{S}_q -module. We show, Theorem 7.2.4, that the image of $\mathcal{U}_{a,\mu}$ under the map ν_n is the set $\Omega_{a,\mu}^{\text{qe}}$ of all spaces C of quasi-exponentials in \mathcal{QE} such that the singularities of C , counted with multiplicity, are encoded by a , and μ encodes the exponents of C at each singular point. See definition 7.2.1 for details.

The space $\text{Hom}_{H_n}(\nabla(q, \mu), \Delta(p, \lambda))$ is a Z_n -module, supported on the intersection of $\Omega_{b,\lambda}$ and $\mathcal{U}_{a,\mu}$. We consider the case $p = 0$ so that $\nu_n(\Omega_{0,\lambda} \cap \mathcal{U}_{a,\mu})$ is contained in $\text{Gr}_n(\mathbb{C}[x]_{2n})$. Set-theoretically, the intersection $\text{Gr}_n(\mathbb{C}[x]_{2n}) \cap \nu_n(\mathcal{U}_{a,\mu})$ is the intersection

$$\Omega_{\mu}(q) = \Omega_{\mu^{(1)}}(q_1) \cap \dots \cap \Omega_{\mu^{(k)}}(q_k)$$

of a certain collection of Schubert cells in $\text{Gr}_n(\mathbb{C}[x]_{2n})$, where the numbers q_i are specifying complete flags in $\mathbb{C}[x]_{2n}$. Under the assumption¹ that we have an equality of (non-reduced) subschemes

$$\text{Gr}_n(\mathbb{C}[x]_{2n}) \cap \nu_n(\mathcal{U}_{a,\mu}) = \Omega_{\mu}(q)$$

of $\text{Gr}_n(\mathbb{C}[x]_{2n})$, we show in Theorem 7.3.4 that

Corollary 1.3.1. *We have an isomorphism of zero-dimensional, Gorenstein schemes*

$$\nu_n : \Omega_{0,\lambda} \cap \mathcal{U}_{a,\mu} \xrightarrow{\sim} \Omega_{0,\lambda^t}^{\text{qe}} \cap \Omega_{\mu}(q)$$

such that $\text{Hom}_{H_n}(\nabla(q, \mu), \Delta(0, \lambda))$ is the coregular (\simeq regular) representation of $\mathbb{C}[\Omega_{0,\lambda} \cap \mathcal{U}_{a,\mu}]$.

We show, independent of the assumption, that

$$\dim \text{Hom}_{H_n}(\nabla(q, \mu), \Delta(0, \lambda)) = \dim \mathbb{C}[\Omega_{0,\lambda^t}^{\text{qe}} \cap \Omega_{\mu}(q)] = \langle \sigma_{\lambda^t}, \sigma_{\mu^{(1)}} \cdots \sigma_{\mu^{(k)}} \rangle,$$

where σ_{\cdot} is the cohomology class in $H^*(\text{Gr}_n(\mathbb{C}[x]_{2n}))$ defined by the closure of a given cell and $\langle -, - \rangle$ is the usual pairing on $H^*(\text{Gr}_n(\mathbb{C}[x]_{2n}))$.

1.4. The results of this article were motivated by recent work of Mukhin, Tarasov and Varchenko. They showed in [27], that there is an intriguing relationship between the rational Cherednik algebra and the Bethe algebra associated to the Gaudin integrable system. Many of the results of this paper were inspired by analogous results of Mukhin, Tarasov and Varchenko about the representation theory of the Bethe algebra.

1.5. **Outline of the article.** In section 2 we recall the definition of rational Cherednik algebras and describe their basic features. We introduce generalized Verma modules.

Section 3 is devoted to the study of the endomorphism ring of generalized Verma modules. It is shown that the centre Z_c of the rational Cherednik algebra H_c surjects onto these endomorphism rings and thus they are commutative. When the Verma module is graded we give the graded character of its endomorphism ring. More generally we study the tor and ext algebras associated to these generalized Verma modules.

Then, in section 4, recall the basic properties (in particular the factorization property) of Wilson's Calogero-Moser space CM_n . Following on from this, we show in section 5 that the isomorphism $\psi_n : X_n \xrightarrow{\sim} \text{CM}_n$ constructed by Etingof and Ginzburg is compatible with the factorizations of X_n and CM_n . We describe, using the representation theory of the degenerate affine Hecke algebra (which is a subalgebra of H_n), the precise bijection between the \mathbb{C}^\times -fixed points of X_n and those of CM_n that is induced by the map ψ_n .

¹See assumption 7.3.2 for details.

Section 6 studies various infinite dimensional Grassmannians and the relationship between them. We show that the various identifications of these different Grassmannian are compatible with the natural factorizations that each space possesses. We also describe the partition of these Grassmannians into Schubert cells and recall the role of the τ -function in describing the embedding of these Grassmannians into projective space.

In the final section we describe the image of the subsets $\Omega_{b,\lambda}$ and $\mathcal{U}_{a,\mu}$ in \mathcal{QE} under the map ν_n described above. We also study the intersections $\Omega_{b,\lambda} \cap \mathcal{U}_{a,\mu}$, giving a proof of Corollary 1.3.1.

The appendix contains a summary of the main results of [1] that are required in the article.

1.6. Conventions. By “ X is an affine variety” we always mean a reduced and irreducible affine scheme of finite type over \mathbb{C} . If the space X is not reduced or not irreducible we call X an affine scheme. We denote by X_{red} the affine variety $\text{Spec}(\mathbb{C}[X]/\text{rad}\mathbb{C}[X])$ corresponding to the affine scheme X . The smooth locus of X will be denoted X_{sm} . When considering a coherent \mathcal{O}_X -module \mathcal{M} , $\text{Supp } \mathcal{M}$ denotes the closed (not necessarily reduced, even if X is reduced) subscheme defined by the annihilator ideal sheaf of \mathcal{M} .

Given a vector space V and $0 < n < \dim V$, we denote by $\text{Gr}_n(V)$ the Grassmannian of all n -dimensional subspaces of V .

If X is a variety with a algebraic \mathbb{C}^\times -action and $x_0 \in X$ a fixed point of this action, then the *attracting set* of x_0 is the set $\{x \in X \mid \lim_{\alpha \rightarrow \infty} \alpha \cdot x = x_0\}$. This convention is chosen so that the coordinate ring of the attracting set is \mathbb{N} -graded.

If M is a \mathbb{Z} -graded module then $M[i]$ is the graded module with $M[i]_j = M_{j-i}$. If each graded piece M_i of M is finite dimensional and $M_i = 0$ for $i \ll 0$ (resp. $i \gg 0$) then we denote by $\text{ch}_q(M)$ the power series

$$\text{ch}_q(M) = \sum_{i \in \mathbb{Z}} \dim M_i q^i$$

in $\mathbb{C}((q))$ (resp. in $\mathbb{C}((q^{-1}))$).

Throughout, \mathbb{N} will denote the natural numbers $\{0, 1, 2, \dots\}$.

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2. RATIONAL CHEREDNIK ALGEBRAS AT $t = 0$

2.1. Definitions and notation. Let (W, \mathfrak{h}) be a complex reflection group, where \mathfrak{h} is the reflection representation for W , and let $\mathcal{S}(W)$ be the set of all complex reflections in W . For each $s \in \mathcal{S}(W)$, choose vectors $\alpha_s \in \mathfrak{h}$ and $\alpha_s^\vee \in \mathfrak{h}^*$ that span the one dimensional spaces $\text{Im}(s-1)|_{\mathfrak{h}}$ and $\text{Im}(s-1)|_{\mathfrak{h}^*}$ respectively. We normalize α_s and α_s^\vee so that $\alpha_s^\vee(\alpha_s) = 2$. Let $\mathbf{c} : \mathcal{S}(W) \rightarrow \mathbb{C}$ be a W -equivariant function. The *rational Cherednik algebra at $t = 0$* , as introduced by Etingof and Ginzburg [12] and denoted $H_{\mathbf{c}}(W)$, is the quotient of the skew group algebra of the tensor algebra $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W$ by the ideal generated by the relations $[x, x'] = [y, y'] = 0$ and

$$[y, x] = \sum_{s \in \mathcal{S}} \mathbf{c}(s) \frac{x(\alpha_s) \alpha_s^\vee(y)}{\alpha_s^\vee(\alpha_s)} s, \quad (2.1.1)$$

for all $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$.

A fundamental result for rational Cherednik algebras is that the PBW property holds for all parameters \mathbf{c} . That is, there is a vector space isomorphism

$$H_{\mathbf{c}}(W) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*]. \quad (2.1.2)$$

2.2. The generalized Calogero-Moser Space. The centre $Z_{\mathbf{c}}(W)$ of $H_{\mathbf{c}}(W)$ is an affine domain over which $H_{\mathbf{c}}(W)$ is a finite module. We shall denote by $X_{\mathbf{c}}(W) := \text{Spec}(Z_{\mathbf{c}}(W))$ the corresponding affine variety. The space $X_{\mathbf{c}}(W)$ is called the *generalized Calogero-Moser space* associated to the complex reflection group W at parameter \mathbf{c} . By [12, Proposition 4.15] we have inclusions $\mathbb{C}[\mathfrak{h}]^W \hookrightarrow Z_{\mathbf{c}}(W)$ and $\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow Z_{\mathbf{c}}(W)$. These inclusions define surjective morphisms $\pi : X_{\mathbf{c}}(W) \rightarrow \mathfrak{h}^*/W$ and $\varpi : X_{\mathbf{c}}(W) \rightarrow \mathfrak{h}/W$ respectively.

Write

$$\Upsilon : X_{\mathbf{c}}(W) \longrightarrow \mathfrak{h}^*/W \times \mathfrak{h}/W$$

for the product morphism $\Upsilon := \pi \times \varpi$. It is a finite, and hence closed, surjective morphism. By putting $x \in \mathfrak{h}^*$ in degree one, $y \in \mathfrak{h}$ in degree -1 and each $w \in W$ in degree zero, it is clear from the relations (2.1.1) that $H_{\mathbf{c}}(W)$ is a \mathbb{Z} -graded algebra. This implies that $Z_{\mathbf{c}}(W)$ is also \mathbb{Z} -graded. Thus, there exists a canonical \mathbb{C}^\times -action on $X_{\mathbf{c}}(W)$. The map Υ is \mathbb{C}^\times -equivariant since $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ is a graded subalgebra of $Z_{\mathbf{c}}(W)$. We recall some of the fundamental properties of $X_{\mathbf{c}}(W)$.

Lemma 2.2.1. *Let $\pi^{-1}(\mathbf{a})$ denote the scheme-theoretic fiber of π over \mathbf{a} , a closed point in \mathfrak{h}^*/W . Then,*

- (1) *The algebra $\mathbb{C}[X_{\mathbf{c}}(W)]$ is free of rank $|W|$ over $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$, hence $X_{\mathbf{c}}(W)$ is Cohen-Macaulay.*
- (2) *The algebra $\mathbb{C}[\pi^{-1}(\mathbf{a})]$ is a free $\mathbb{C}[\mathfrak{h}]^W$ -module of rank $|W|$.*

Proof. By [12, Proposition 4.15], $Z_{\mathbf{c}}(W)$ is a free $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ -module of rank $|W|$. Therefore $\mathbb{C}[\pi^{-1}(\mathbf{a})]$ is a free $\mathbb{C}[\mathfrak{h}]^W$ -module of rank $|W|$. \square

The affine scheme $\pi^{-1}(\mathbf{a})$ is neither reduced nor irreducible. The generalized Calogero-Moser space $X_{\mathbf{c}}(W)$ has a natural Poisson structure, see [12].

2.3. $H_{\mathbf{c}}(W)$ -modules and fixed points of the \mathbb{C}^\times -action. We define those $H_{\mathbf{c}}(W)$ -modules that will be the focus of the rest of the article.

Definition 2.3.1. Choose $p \in \mathfrak{h}^*$, $q \in \mathfrak{h}$, $\mathbf{a} \in \mathfrak{h}/W$ and $\mathbf{b} \in \mathfrak{h}^*/W$. Let the stabilizer of p (resp. of q) be denoted by W_p (resp. W_q)

- (1) Choose $\lambda \in \text{Irr}(W_p)$. The *generalized Verma module* associated to p and λ is defined to be

$$\Delta(p, \lambda) := H_{\mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}^*] \rtimes W_p} \lambda,$$

where the action of $\mathbb{C}[\mathfrak{h}^*]$ on λ is via evaluation at p . When $p = 0$, $\Delta(p, \lambda)$ will be written $\Delta(\lambda)$ and is called a Verma module.

- (2) Let $\mathfrak{m}_{\mathbf{a}} \triangleleft \mathbb{C}[\mathfrak{h}]^W$ the maximal ideal corresponding to \mathbf{a} . The *generalized baby Verma module* associated to p , λ and \mathbf{a} is defined to be

$$\Delta(p, \lambda, \mathbf{a}) := \Delta(p, \lambda) / \mathfrak{m}_{\mathbf{a}} \cdot \Delta(p, \lambda).$$

- (3) Choose $\mu \in \text{Irr}(W_q)$. The *dual generalized dual Verma module* associated to q and μ is defined to be

$$\nabla(q, \mu) := H_{\mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}] \rtimes W_q} \mu.$$

where the action of $\mathbb{C}[\mathfrak{h}]$ on μ is via evaluation at q .

- (4) Let $\mathfrak{n}_{\mathbf{b}} \triangleleft \mathbb{C}[\mathfrak{h}^*]^W$ the maximal ideal corresponding to \mathbf{b} . The *dual generalized baby Verma module* associated to q , μ and \mathbf{b} is defined to be

$$\nabla(q, \mu, \mathbf{b}) := \nabla(q, \mu) / \mathfrak{n}_{\mathbf{b}} \cdot \nabla(q, \mu).$$

The module $\Delta(0, \lambda, 0)$ is the baby Verma module studied in [18]. Motivated by the connection between the Calogero-Moser space and certain ordinary differential equations:

Definition 2.3.2. A simple $H_c(\mathfrak{h}, W)$ -module L is said to be *Fuchsian* if $\mathbb{C}[\mathfrak{h}^*]_+^W \cdot L = 0$, where $\mathbb{C}[\mathfrak{h}^*]_+^W$ is the augmentation ideal of $\mathbb{C}[\mathfrak{h}^*]$.

The coinvariant ring $\mathbb{C}[\mathfrak{h}]^{coW}$ is defined to be the finite dimensional quotient $\mathbb{C}[\mathfrak{h}]/\langle \mathbb{C}[\mathfrak{h}]_+^W \rangle$ of $\mathbb{C}[\mathfrak{h}]$, where $\langle \mathbb{C}[\mathfrak{h}]_+^W \rangle$ is the ideal in $\mathbb{C}[\mathfrak{h}]$ generated by the augmentation ideal of $\mathbb{C}[\mathfrak{h}]^W$. Since W is a complex reflection group, the coinvariant ring $\mathbb{C}[\mathfrak{h}]^{coW}$ is isomorphic to the regular representation as a W -module. Choose $p \in \mathfrak{h}^*$, $q \in \mathfrak{h}$, $\mathbf{a} \in \mathfrak{h}/W$ and $\mathbf{b} \in \mathfrak{h}^*/W$ as in definition 2.3.1. The image of p in \mathfrak{h}^*/W is denoted \bar{p} and similarly for q . Let δ be the determinant character of W_q i.e. $\delta(w) = \det(w|_{\mathfrak{h}})$ for all $w \in W_q$.

Lemma 2.3.3. *If $\bar{p} = \mathbf{b}$ and $\bar{q} = \mathbf{a}$ then*

$$\dim \text{Hom}_{H_c(W)}(\nabla(q, \mu, \mathbf{b}), \Delta(p, \lambda, \mathbf{a})) = \dim \text{Hom}_W(\text{Ind}_{W_q}^W(\mu \otimes \delta^*), \text{Ind}_{W_p}^W \lambda);$$

otherwise $\text{Hom}_{H_c(W)}(\nabla(q, \mu, \mathbf{b}), \Delta(p, \lambda, \mathbf{a})) = 0$.

Proof. If $\bar{p} \neq \mathbf{b}$, then $\mathbf{n}_{\mathbf{b}} \cdot \Delta(p, \lambda, \mathbf{a}) \neq 0$ and hence $\text{Hom}_{H_c(W)}(\nabla(q, \mu, \mathbf{b}), \Delta(p, \lambda, \mathbf{a})) = 0$. Therefore we assume that $\bar{p} = \mathbf{b}$. Then,

$$\begin{aligned} \text{Hom}_{H_c(W)}(\nabla(q, \mu, \mathbf{b}), \Delta(p, \lambda, \mathbf{a})) &= \text{Hom}_{H_c(W)}(\nabla(q, \mu), \Delta(p, \lambda, \mathbf{a})) \\ &= \text{Hom}_{\mathbb{C}[\mathfrak{h}] \rtimes W_q}(\mu, \Delta(p, \lambda, \mathbf{a})). \end{aligned}$$

As a $\mathbb{C}[\mathfrak{h}]$ -module, μ is just the direct sum of $\dim \mu$ copies of the skyscraper sheaf at q . If a_1, \dots, a_k are the points in the W_p -orbit corresponding to \mathbf{a} , then

$$\Delta(p, \lambda, \mathbf{a}) = \bigoplus_{i=1}^k \mathbb{C}[\mathfrak{h}]_{a_i} \otimes \text{Ind}_{W_p}^W \lambda$$

as a $\mathbb{C}[\mathfrak{h}]$ -module, where $\mathbb{C}[\mathfrak{h}]_{a_i}$ is a module supported at a_i . Then,

$$\text{Hom}_{\mathbb{C}[\mathfrak{h}] \rtimes W_q}(\mu, \Delta(p, \lambda, \mathbf{a})) \subset \text{Hom}_{\mathbb{C}[\mathfrak{h}]}(\mu, \Delta(p, \lambda, \mathbf{a}))$$

implies that $\text{Hom}_{\mathbb{C}[\mathfrak{h}] \rtimes W_q}(\mu, \Delta(p, \lambda, \mathbf{a})) = 0$ unless $\bar{q} = \mathbf{a}$. Therefore, we assume that $\bar{q} = \mathbf{a}$. Then, there exists some i such that $q = a_i$ and

$$\text{Hom}_{\mathbb{C}[\mathfrak{h}] \rtimes W_q}(\mu, \Delta(p, \lambda, \mathbf{a})) = \text{Hom}_{\mathbb{C}[\mathfrak{h}] \rtimes W_q}(\mu, \mathbb{C}[\mathfrak{h}]_q \otimes \text{Ind}_{W_p}^W \lambda).$$

Under the automorphism $x \mapsto x - q(x)$ of $\mathbb{C}[\mathfrak{h}] \rtimes W_q$, the module $\mathbb{C}[\mathfrak{h}]_q$ is sent to $\mathbb{C}[\mathfrak{h}]^{coW_q}$ and μ is sent to μ_0 , which is defined to be the $\mathbb{C}[\mathfrak{h}] \rtimes W_q$ -module isomorphic to μ as a W_q -module and supported at 0. Thus,

$$\text{Hom}_{\mathbb{C}[\mathfrak{h}] \rtimes W_q}(\mu, \mathbb{C}[\mathfrak{h}]_q \otimes \text{Ind}_{W_p}^W \lambda) \simeq \text{Hom}_{\mathbb{C}[\mathfrak{h}] \rtimes W_q}(\mu_0, \mathbb{C}[\mathfrak{h}]^{coW_q} \otimes \text{Ind}_{W_p}^W \lambda) \quad (2.3.4)$$

$$= \text{Hom}_{\mathbb{C}[\mathfrak{h}] \rtimes W_q}(\mu_0, \text{soc}(\mathbb{C}[\mathfrak{h}]^{coW_q}) \otimes \text{Ind}_{W_p}^W \lambda). \quad (2.3.5)$$

The socle of $\mathbb{C}[\mathfrak{h}]^{coW_q}$ is a W_q -module. It is known that this W_q -module is the determinant character δ , see [24, Corollary 4.23 (iii)]. Therefore, the space (2.3.5) can be identified with

$$\text{Hom}_{W_q}(\mu, \delta \otimes \text{Ind}_{W_p}^W \lambda) \simeq \text{Hom}_W(\text{Ind}_{W_q}^W(\mu \otimes \delta^*), \text{Ind}_{W_p}^W \lambda).$$

□

2.4. Since $H_c(W)$ is a finite module over its centre, each simple $H_c(W)$ -module L is finite dimensional. Moreover, by [12, Theorem 1.7], we have $\dim L \leq |W|$ with equality if and only if the support of L , which is a closed point of $X_c(W)$, is contained in the smooth locus. As noted above, the map Υ is \mathbb{C}^\times -equivariant. Since the image of 0 in $\mathfrak{h}^*/W \times \mathfrak{h}/W$ is the unique \mathbb{C}^\times -fixed point of that space, the finitely many closed point of $\Upsilon^{-1}(0)$ are the precisely the \mathbb{C}^\times -fixed points in $X_c(W)$. The simple $H_c(W)$ -modules supported at each of these fixed points is a graded $H_c(W)$ -module. These simple modules are modules for the *restricted rational Cherednik algebra* $\overline{H}_c(W)$, which is the quotient of $H_c(W)$ by the ideal generated by $\mathbb{C}[\mathfrak{h}]_+^W$ and $\mathbb{C}[\mathfrak{h}^*]_+^W$.

It is shown in [18, Proposition 4.3] that these graded, simple modules are naturally parameterized by the set $\text{Irr}(W)$. We see from the definition of $\overline{H}_c(W)$ that each of the baby Verma modules $\Delta(0, \lambda, 0)$ is a $\overline{H}_c(W)$ -module. Then, for each $\lambda \in \text{Irr}(W)$, the head of $\Delta(0, \lambda, 0)$ is a simple, graded $\overline{H}_c(W)$ -module, denoted $L(\lambda)$. The support of $L(\lambda)$, a closed point in $X_c(W)$, will be denoted x_λ . Thus, we have a surjective map $\text{Irr}(W) \rightarrow \{\mathbb{C}^\times\text{-fixed points in } X_c(W)\}$, which is a bijection if and only if $X_c(W)$ is smooth.

3. ENDOMORPHISM ALGEBRAS

3.1. Let W be a complex reflection group and fix $\lambda \in \text{Irr}(W)$. In the first part of this section we consider the Verma modules $\Delta(\lambda)$, which are supported on $\pi^{-1}(0)$. These modules are \mathbb{Z} -graded, a fact that will be crucial for our arguments.

Lemma 3.1.1. *If the \mathbb{C}^\times -fixed point x_λ is contained in the smooth locus of $X_c(W)$ then $\text{Supp } \Delta(\lambda)$ is contained in the smooth locus of $X_c(W)$.*

Proof. The singular locus of $X_c(W)$ is \mathbb{C}^\times -stable. The module $\Delta(\lambda)$ is graded by putting $1 \otimes \lambda$ in degree zero. Then, all non-zero weight spaces are positive. Therefore, if I is the annihilator of $\Delta(\lambda)$ in $Z_c(W)$, then the quotient $Z_c(W)/I$ is positively graded with degree zero part equal to \mathbb{C} . This implies that $\text{Supp } \Delta(\lambda)$ is contained inside the attracting set $\{x \in X_c(W) \mid \lim_{t \rightarrow \infty} t \cdot x = x_\lambda\}$. The singular locus $X_c(W)_{\text{sing}}$ of $X_c(W)$ is \mathbb{C}^\times -stable. Therefore, if $\text{Supp } \Delta(\lambda) \cap X_c(W)_{\text{sing}} \neq \emptyset$ then x_λ belongs to this intersection. \square

For $\mu \in \text{Irr}(W)$, the polynomial that records the graded multiplicity of μ in $\mathbb{C}[\mathfrak{h}]^{coW}$ is called the *fake polynomial* $f_\mu(t)$ of μ . The degree of the lowest monomial appearing in $f_\mu(t)$ is denoted b_μ .

Lemma 3.1.2. *If the \mathbb{C}^\times -fixed point x_λ is contained in the smooth locus of $X_c(W)$ then the endomorphism ring $E_{(\lambda,0)} := \text{End}_{H_c(W)}(\Delta(0, \lambda, 0))$ of the baby Verma module $\Delta(0, \lambda, 0)$ is \mathbb{N} -graded, commutative and $Z_c(W)$ surjects onto $E_{(\lambda,0)}$. In this case, the graded character of $E_{(\lambda,0)}$ is given by*

$$\text{ch}_q(E_{(\lambda,0)}) = q^{-b_\lambda} f_{\lambda*}(q).$$

Proof. Since $\Delta(0, \lambda, 0)$ is indecomposable, it belongs to a single block of the restricted rational Cherednik algebra $\overline{H}_c(W)$. Since we have assumed that the fixed point of $X_c(W)$ labeled by λ is contained in the smooth locus, this block of the restricted rational Cherednik algebra is isomorphic to $\text{Mat}_{|W|}(A)$, where A is some finite dimensional, local, graded quotient of $Z_c(W)$ - see [18, §5.3]. If $P(\lambda)$ is the projective cover of $L(\lambda)$ in $\overline{H}_c(W)\text{-mod}$ then $P(\lambda) = V(A)$, the “vectorial” representation of $\text{Mat}_{|W|}(A)$. Since $L(\lambda)$ is a quotient of the indecomposable module $\Delta(0, \lambda, 0)$, the module $\Delta(0, \lambda, 0)$ is a quotient of $P(\lambda)$. Hence $\Delta(0, \lambda, 0) = V(E_{(\lambda,0)})$ for some A -module $E_{(\lambda,0)}$. Clearly $E_{(\lambda,0)}$ is actually a graded quotient of A and hence also a quotient of $Z_c(W)$. Let L be the unique simple module for A , considered as a graded module concentrated in degree zero so that the natural map $A \twoheadrightarrow L$ is graded of degree zero. Then

$$\text{ch}_q(E_{(\lambda,0)}) = \sum_{i \geq 0} (E_{(\lambda,0)} : L[i]) q^i = \sum_{i \geq 0} (\Delta(0, \lambda, 0) : L(\lambda)[i]) q^i.$$

The proof of [18, Theorem 5.6] shows that $\sum_{i \geq 0} (\Delta(0, \lambda, 0) : L(\lambda)[i]) q^i$ equals $q^{-b_{\lambda^*}} f_{\lambda^*}(q) \in \mathbb{N}[q]$. \square

Recall that the *degrees* of a complex reflection group (W, \mathfrak{h}) is the multiset of degrees of some (any) choice of homogeneous, algebraically independant generators of $\mathbb{C}[\mathfrak{h}]^W$.

Theorem 3.1.3. *Assume that the \mathbb{C}^\times -fixed point \mathfrak{x}_λ is contained in the smooth locus of $X_c(W)$. Then,*

- (1) *The endomorphism ring E_λ of the Verma module $\Delta(\lambda)$ is an \mathbb{N} -graded quotient of $Z_c(W)$. In particular, it is commutative.*
- (2) *$\Delta(\lambda)$ is a cyclic E_λ -module.*
- (3) *The graded character of E_λ is*

$$\text{ch}_q(E_\lambda) = q^{-b_{\lambda^*}} f_{\lambda^*}(q) \prod_{i=1}^n (1 - q^{d_i})^{-1},$$

where d_1, \dots, d_n are the degrees of (W, \mathfrak{h}) .

Proof. Throughout the proof, a graded map or graded morphism will mean a graded map of degree zero. Since $\Delta(\lambda)_i = 0$ for all $i < 0$, E_λ must be an \mathbb{N} -graded algebra. We begin by giving an upper bound on its graded character and then show that the graded character of the image of the centre of $H_c(W)$ in E_λ is at least as big. Since $\Delta(\lambda)$ is generated by a copy of the irreducible W -module λ , lying in degree zero, it has a unique maximal graded submodule M and the quotient is $L(\lambda)$. The grading on $L(\lambda)$ is defined so that the quotient map $\Delta(\lambda) \rightarrow L(\lambda)$ is graded. Under our assumption, the module $L(\lambda)$ is isomorphic to $\mathbb{C}W$ as a $\mathbb{C}W$ -module, thus there exists a unique copy of the trivial $\mathbb{C}W$ -module $\mathbf{1}$ in $L(\lambda)$. There is a unique graded copy $\hat{\mathbf{1}}$ of the trivial W -representation in $\Delta(\lambda)$ mapping onto $\mathbf{1}$. Now choose $\phi \in E_\lambda$ a nonzero, homogeneous element of degree k . Then, $\phi(1 \otimes \lambda) \neq 0$ and, since ϕ is homogeneous, $\phi(M) \subsetneq \phi(\Delta(\lambda))$ implies that $\phi(\hat{\mathbf{1}}) \neq 0$. If $\deg(\hat{\mathbf{1}}) = a$ then $\phi \mapsto \phi(\hat{\mathbf{1}})[-a]$ defines an injective, graded map of vector spaces $E_\lambda \rightarrow e\Delta(\lambda)[-a]$. As in the proof of [18, Theorem 5.6], a direct calculation shows that $a = b_{\lambda^*}$ and

$$\text{ch}_q(e\Delta(\lambda)[-b_{\lambda^*}]) = q^{-b_{\lambda^*}} f_{\lambda^*}(q) \prod_{i=1}^n (1 - q^{d_i})^{-1}.$$

By Lemma 3.1.2, $E_{(\lambda,0)}$ is a graded quotient of $Z_c(W)$. We fix a graded lift \tilde{E}_λ of $E_{(\lambda,0)}$ in $Z_c(W)$ as vector spaces. Define by multiplication the graded map

$$\tau : \tilde{E}_\lambda \otimes \mathbb{C}[\mathfrak{h}]^W \longrightarrow E_\lambda.$$

This map factors through $Z_c(W)$. The graded character of $\tilde{E}_\lambda \otimes \mathbb{C}[\mathfrak{h}]^W$ is also $q^{-b_{\lambda^*}} f_{\lambda^*}(q) \prod_{i=1}^n (1 - q^{d_i})^{-1}$. Therefore, the result will follow if we can show that τ is injective. Let $a = \sum_{j=1}^m a_j \otimes f_j \in \tilde{E}_\lambda \otimes \mathbb{C}[\mathfrak{h}]^W$ such that $\tau(a) = 0$. We may assume without loss of generality that a is homogeneous. Let $k \geq 0$ be the smallest integer such that $\deg(f_i) \geq k$ for all i and \mathfrak{m}_k the ideal in $\mathbb{C}[\mathfrak{h}]^W$ of all element of degree greater than k . Set $\Delta_k(\lambda) = \Delta(\lambda)/\mathfrak{m}_k \Delta(\lambda)$, then $\tau(a)$ is also an endomorphism of $\Delta_k(\lambda)$. By assumption, this endomorphism is zero. We may now assume that $\{f_1, \dots, f_m\}$ is in fact a basis of $\mathbb{C}[\mathfrak{h}]_k^W$, the space of all poynomials of degree k . We have

$$\mathfrak{m}_{k-1} \Delta_k(\lambda) \simeq \Delta(0, \lambda, 0)^{\oplus m}.$$

The subspace $\mathfrak{m}_0 \Delta_k(\lambda)$ is contained in the kernel of τ and the image of τ is contained in $\mathfrak{m}_{k-1} \Delta_k(\lambda)$. Thus, τ induces a map

$$\tilde{\tau} : \Delta(0, \lambda, 0) \simeq \Delta_k(\lambda)/\mathfrak{m}_0 \Delta_k(\lambda) \longrightarrow \Delta(0, \lambda, 0)^{\oplus m}.$$

The map $\tilde{\tau}$ is just $(\bar{a}_1, \dots, \bar{a}_m)$, where \bar{a}_i is the image of a_i in $E_{(\lambda,0)}$. Thus, $\tilde{\tau}$ is zero if and only if each a_i is zero. Hence $\tilde{\tau}$ implies that $a = 0$. \square

Remark 3.1.4. Lemma 3.1.2 and Theorem 3.1.3 are false if the assumption on x_λ is dropped.

Corollary 3.1.5. *The commutative ring E_λ is a graded polynomial.*

Proof. The algebra E_λ is \mathbb{N} -graded and connected. Therefore, there is a unique graded, maximal ideal \mathfrak{m} in E_λ . The algebra E_λ is a finite module over $\mathbb{C}[\mathfrak{h}]^W$. Therefore, by [4, Corollary 1.4.5], the Krull dimension of E_λ is $\dim \mathfrak{h}$. Let $T = (E_+/E_+^2)^*$ be the tangent space at the unique closed \mathbb{C}^\times -fixed point of $\text{Spec}(E_\lambda)$. If we can show that $\dim T = n$ then the corollary follows from the well-known claim:

Claim 3.1.6. Let R be an \mathbb{N} -graded, commutative \mathbb{C} -algebra such that $R/R_+ = \mathbb{C}$ and R has Krull dimension n . If the tangent space $U = (R_+/R_+^2)^*$ has dimension n then we have an isomorphism of graded algebras $\mathbb{C}[U] \xrightarrow{\sim} R$.

Let's show $\dim T = n$. It must have dimension at least n . Since $\text{Spec}(E_\lambda)$ is a closed subvariety of $X_c(W)$, T is a subspace of $T_{x_\lambda} X_c(W)$. The fact that x_λ is a \mathbb{C}^\times -fixed point implies that the space $T_{x_\lambda} X_c(W)$ is a \mathbb{C}^\times -module and T is a submodule. We decompose $T_{x_\lambda} X_c(W) = T_- \oplus T_0 \oplus T_+$, where T_- consists of all weight spaces of strictly negative weights etc. We have $T \subset T_+$. Since x_λ is an isolated fixed point, it follows from [6, Corollary 2.2] that $T_0 = 0$. By assumption, x_λ is contained in the smooth locus of $X_c(W)$. Therefore, $T_{x_\lambda} X_c(W)$ is a symplectic vector space. The symplectic form on $T_{x_\lambda} X_c(W)$ is \mathbb{C}^\times -invariant. This implies that $\dim T_+ = \dim T_- = n$. Thus, $T \subset T_+$ implies that $\dim T \leq n$ as required. \square

Corollary 3.1.7. *The E_λ -module $\Delta(\lambda)$ is free of rank $|W|$.*

Proof. Since we have shown in Corollary 3.1.5 that E_λ is a polynomial ring, and $\Delta(\lambda)$ is a finitely generated E_λ -module, it suffices to show that $\Delta(\lambda)$ is projective, or equivalently, locally free. Let $\mathfrak{m} \subset E_\lambda$ be a maximal ideal and consider the quotient $L = \Delta(\lambda)/\mathfrak{m} \cdot \Delta(\lambda)$. This space is non-zero since any endomorphism $\phi \in E_\lambda$ whose image in $E_\lambda/\mathfrak{m} \cdot E_\lambda$ is non-zero induces a non-zero endomorphism of L . On the other hand, the proof of Theorem 3.1.3 shows that the map

$$E_\lambda/\mathfrak{m} \cdot E_\lambda \longrightarrow e\Delta(\lambda)/\mathfrak{m}e\Delta(\lambda) = eL$$

is an isomorphism. In particular, $\dim eL = 1$. By Lemma 3.1.1, the support of $\Delta(\lambda)$, and hence of L too, is contained in the smooth locus of $X_c(W)$. Therefore, L is simple and has dimension $|W|$. Thus, every fiber of $\Delta(\lambda)$ has dimension $|W|$ over E_λ . \square

3.2. Now we will consider homomorphisms between the various $\Delta(\lambda)$.

Lemma 3.2.1. *Let $\lambda \neq \mu \in \text{Irr}(W)$. If either of x_λ or x_μ belongs to the smooth locus of $X_c(W)$ then*

$$\text{Hom}_{H_c(W)}(\Delta(\lambda), \Delta(\mu)) = 0$$

Proof. Without loss of generality, x_λ is contained in the smooth locus of $X_c(W)$. We begin by remarking that this implies that the block of the restricted rational Cherednik algebra $\overline{H}_c(W)$ containing $L(\lambda)$ is a singleton. In particular, $\text{Hom}_{\overline{H}_c(W)}(\Delta(0, \lambda, 0), \Delta(0, \mu, 0)) = 0$.

Let $\mathfrak{m}_i \subset \mathbb{C}[\mathfrak{h}]^W$ be the subspace of all elements of degree at least i . It is an ideal in $\mathbb{C}[\mathfrak{h}]^W$. We set $\Delta_i(\lambda) = \mathfrak{m}_i \Delta(\lambda)$, a graded submodule of $\Delta(\lambda)$. This defines a filtration of $\Delta(\lambda)$ such that

$$\Delta_i(\lambda)/\Delta_{i+1}(\lambda) \simeq (\Delta(0, \lambda, 0)[-i])^{\oplus N_i},$$

as a graded $H_c(W)$ -module, where $N_i = \dim \mathfrak{m}_i/\mathfrak{m}_{i+1}$.

Let $0 \neq \phi \in \text{Hom}_{H_c(W)}(\Delta(\lambda), \Delta(\mu))$. Since the space $\text{Hom}_{H_c(W)}(\Delta(\lambda), \Delta(\mu))$ is graded, we may assume without loss of generality that ϕ is homogeneous. The fact that $\Delta(\lambda)$ is generated by $1 \otimes \lambda$ implies that $M = \phi(1 \otimes \lambda)$ is a non-zero, graded subspace of $\Delta(\mu)$ that generates the image of

ϕ . Let i be the largest integer such that $M \subset \Delta_i(\mu)$ so that the image of M in $\Delta_i(\mu)/\Delta_{i+1}(\mu)$ is non-zero. Then, ϕ descends to a non-zero morphism

$$\tilde{\phi} : \Delta(0, \lambda, 0) \rightarrow (\Delta(0, \mu, 0)[-i])^{\oplus N_i}.$$

This implies that $\text{Hom}_{\text{Hc}(W)}(\Delta(0, \lambda, 0), \Delta(0, \mu, 0))$ is non-zero, which contradicts our initial assumption. \square

We have shown in Theorem 3.1.3 that $Z_{\mathbf{c}}(W)$ surjects on to $\text{End}_{\text{Hc}}(\Delta(\lambda))$. Let K be the ideal in $Z_{\mathbf{c}}(W)$ defining $\pi^{-1}(0)_{\text{red}}$.

Proposition 3.2.2. *Assume that $X_{\mathbf{c}}(W)$ is smooth. Then, multiplication defines a graded isomorphism*

$$Z_{\mathbf{c}}(W)/K \xrightarrow{\sim} \text{End}_{\text{Hc}(W)} \left(\bigoplus_{\lambda \in \text{Irr}(W)} \Delta(\lambda) \right). \quad (3.2.3)$$

Proof. Lemma 3.2.1 shows that $\text{End}_{\text{Hc}(W)} \left(\bigoplus_{\lambda \in \text{Irr}(W)} \Delta(\lambda) \right) \simeq \bigoplus_{\lambda \in \text{Irr}(W)} \text{End}_{\text{Hc}(W)}(\Delta(\lambda))$. Theorem 3.1.3 (1) together with Corollary 3.1.5 implies that the support of $\Delta(\lambda)$ is reduced. Thus, $K \cdot \Delta(\lambda) = 0$ for all $\lambda \in \text{Irr}(W)$. The supports of the modules $\Delta(\lambda)$ are also disjoint because they are precisely the attracting sets for the \mathbb{C}^\times -action. These facts imply the statement of the proposition. \square

Remark 3.2.4. Even when $X_{\mathbf{c}}(W)$ is smooth, one can show that $\pi^{-1}(0)$ is not reduced. In fact, it will be reduced if and only if $W \simeq \mathbb{Z}_m$ and \mathbf{c} generic. To see this, notice that Theorem 3.2.2 implies that $Z_{\mathbf{c}}(W)/K$ is positively graded. However, if W has irreducible representations of dimension greater than one then $Z_{\mathbf{c}}(W)/\langle \mathbb{C}[\mathfrak{h}^*]_+^W \rangle$ has non-zero graded pieces of negative degree.

3.3. Generalized Verma modules. In this section we extend Theorem 3.1.3 to generalized Verma modules. This is done by showing that the endomorphism ring of $\Delta(p, \lambda)$ is isomorphic to the endomorphism ring of the $\text{H}_{\mathbf{c}'}(\mathfrak{h}, W_p)$ -module $\Delta(\lambda)$, where \mathbf{c}' denotes the restriction of \mathbf{c} to the reflections in W_p . First, we recall a certain completion of $\text{H}_{\mathbf{c}}(\mathfrak{h}, W)$, as defined by Bezrukavnikov and Etingof [5]. Our presentation will be slightly different to *loc. cit.* because we require it to agree with Wilson's factorization of the Calogero-Moser space when W is the symmetric group.

Let $p \in \mathfrak{h}^*$ and let \mathbf{b} be the image of p in \mathfrak{h}^*/W . We denote by $\mathfrak{m}_{\mathbf{b}}$ for the maximal ideal of $\mathbb{C}[\mathfrak{h}^*]^W$ corresponding to \mathbf{b} . The completion of $\text{H}_{\mathbf{c}}(\mathfrak{h}, W)$ with respect to the ideal generated by $\mathfrak{m}_{\mathbf{b}}$ is denoted $\widehat{\text{H}}_{\mathbf{c}}(\mathfrak{h}, W)_{\mathbf{b}}$. The algebra $\mathbb{C}[\mathfrak{h}^*]/\mathfrak{m}_{\mathbf{b}} \cdot \mathbb{C}[\mathfrak{h}^*]$ is finite dimensional with closed points corresponding to the W -orbit of p . Assume this orbit consists of l points $p_1 = p, p_2, \dots, p_l$. Then, there exist primitive idempotents $\tilde{e}_1, \dots, \tilde{e}_l \in \mathbb{C}[\mathfrak{h}^*]/\mathfrak{m}_{\mathbf{b}} \cdot \mathbb{C}[\mathfrak{h}^*]$ such that

$$\mathbb{C}[\mathfrak{h}^*]/\mathfrak{m}_{\mathbf{b}} \cdot \mathbb{C}[\mathfrak{h}^*] = \bigoplus_{i=1}^l (\mathbb{C}[\mathfrak{h}^*]/\mathfrak{m}_{\mathbf{b}} \cdot \mathbb{C}[\mathfrak{h}^*]) \cdot \tilde{e}_i.$$

Hensel's Lemma, [10, Corollary 7.5], implies that the primitive idempotents in

$$\widehat{\mathbb{C}[\mathfrak{h}^*]}_{\mathbf{b}} := \varprojlim_k \mathbb{C}[\mathfrak{h}^*]/\mathfrak{m}_{\mathbf{b}}^k \cdot \mathbb{C}[\mathfrak{h}^*]$$

are precisely the lifts e_i of the \tilde{e}_i . Therefore,

$$\widehat{\mathbb{C}[\mathfrak{h}^*]}_{\mathbf{b}} = \bigoplus_{i=1}^l \widehat{\mathbb{C}[\mathfrak{h}^*]}_{\mathbf{b}} \cdot e_i = \bigoplus_{i=1}^l \widehat{\mathbb{C}[\mathfrak{h}^*]}_{p_i}$$

with each $\widehat{\mathbb{C}[\mathfrak{h}^*]}_{\mathfrak{p}} \cdot e_i$ a local ring. For all $k \geq 0$, the PBW-Theorem (2.1.2) implies that

$$\mathbb{C}[\mathfrak{h}^*] \cap \mathfrak{m}_{\mathfrak{b}}^k \cdot H_{\mathfrak{c}}(W, \mathfrak{h}) = \mathfrak{m}_{\mathfrak{b}}^k \cdot \mathbb{C}[\mathfrak{h}^*].$$

Hence, we have embeddings

$$\mathbb{C}[\mathfrak{h}^*]/\mathfrak{m}_{\mathfrak{b}}^k \cdot \mathbb{C}[\mathfrak{h}^*] \hookrightarrow H_{\mathfrak{c}}(W, \mathfrak{h})/\mathfrak{m}_{\mathfrak{b}}^k \cdot H_{\mathfrak{c}}(W, \mathfrak{h})$$

and taking the inductive limit, $\widehat{\mathbb{C}[\mathfrak{h}^*]}_{\mathfrak{b}} \hookrightarrow \widehat{H}_{\mathfrak{c}}(W, \mathfrak{h})_{\mathfrak{b}}$, where we have used the fact that the functor of inverse limit is left exact. Therefore, we have $e_i \in \widehat{H}_{\mathfrak{c}}(W, \mathfrak{h})_{\mathfrak{b}}$ for all i and

$$\widehat{H}_{\mathfrak{c}}(W, \mathfrak{h})_{\mathfrak{b}} = \bigoplus_{i,j=1}^l e_i \widehat{H}_{\mathfrak{c}}(W, \mathfrak{h})_{\mathfrak{b}} e_j. \quad (3.3.1)$$

Let $\widehat{H}_{\mathfrak{c}'}(W_{p_i}, \mathfrak{h})_{p_i}$ denote the completion of $H_{\mathfrak{c}'}(W_{p_i}, \mathfrak{h})$ with respect to the maximal ideal \mathfrak{n}_{p_i} in $\mathbb{C}[\mathfrak{h}^*]^{W_{p_i}}$. We write $\widehat{\mathbb{C}[\mathfrak{h}^*]}_{p_i}$ for the completion of $\mathbb{C}[\mathfrak{h}^*]$ with respect to the maximal ideal \mathfrak{n}_{p_i} in order to distinguish it from $\widehat{\mathbb{C}[\mathfrak{h}^*]}_{p_i}$ (though these rings are isomorphic by Lemma 3.3.2). Write also $W_{i,j}$ for the subset of W consisting of all elements w such that $w \cdot e_i = e_j$. Before proving the main result of the section we require a series of preparatory lemmata.

Lemma 3.3.2. *Let R be a commutative Noetherian ring and I an ideal of R . Then, the natural map*

$$\varprojlim R/I^k \rightarrow \varprojlim R/(\text{rad } I)^k$$

is an isomorphism, where $\text{rad } I$ is the radical of I .

Proof. The result [10, Lemma 7.14] says that the map will be an isomorphism if for each k there exists some l such that $I^l \subset (\text{rad } I)^k$ and some m such that $(\text{rad } I)^m \subset I^k$. Since $I \subset \text{rad } I$ we have $I^k \subset (\text{rad } I)^k$ for all k . So we just need to find some m such that $(\text{rad } I)^m \subset I^k$. Since R is Noetherian, both I and $\text{rad } I$ are finitely generated. Fix generators f_1, \dots, f_r of $\text{rad } I$ and let $n_i \in \mathbb{N}$ such that $f_i^{n_i} \in I$. If we take $m = 1 + \prod_{i=1}^r (kn_i - 1)$ then

$$(a_1 f_1 + \dots + a_r f_r)^m = \sum_{\mathbf{i}=i_1, \dots, i_m} f_{i_1} \dots f_{i_m} a_{\mathbf{i}},$$

for some $a_{\mathbf{i}} \in R$. By our choice of m , each product $f_{i_1} \dots f_{i_m}$ contains some f_i at least $n_i k$ times. Therefore, $f_{i_1} \dots f_{i_m} a_{\mathbf{i}} \in I^k$, as required. \square

Lemma 3.3.3. *Multiplication defines a vector space isomorphism*

$$e_i(\mathbb{C}W_{i,j} \otimes \mathbb{C}[\mathfrak{h}]) \widehat{\otimes} \widehat{\mathbb{C}[\mathfrak{h}^*]}_{p_j} e_j \xrightarrow{\sim} e_i \widehat{H}_{\mathfrak{c}}(W, \mathfrak{h})_{\mathfrak{b}} e_j.$$

and hence

$$e_i \widehat{H}_{\mathfrak{c}}(W, \mathfrak{h})_{\mathfrak{b}} e_j \otimes e_j \widehat{H}_{\mathfrak{c}}(W, \mathfrak{h})_{\mathfrak{b}} e_k \rightarrow e_i \widehat{H}_{\mathfrak{c}}(W, \mathfrak{h})_{\mathfrak{b}} e_k$$

for all $1 \leq i, j, k \leq l$.

Proof. Lemma 3.3.2 implies that we have canonical isomorphisms

$$\widehat{\mathbb{C}[\mathfrak{h}^*]}_{\mathfrak{b}} \cdot e_j \simeq \widehat{\mathbb{C}[\mathfrak{h}^*]}_{p_j} \simeq \widehat{\mathbb{C}[\mathfrak{h}^*]}_{p_j}.$$

Therefore, multiplication is a well-defined map. The algebra $\widehat{H}_{\mathfrak{c}}(W, \mathfrak{h})_{\mathfrak{b}}$ is filtered by putting $\mathbb{C}W \otimes \widehat{\mathbb{C}[\mathfrak{h}^*]}_{\mathfrak{b}}$ in degree zero and $\mathfrak{h}^* \subset \mathbb{C}[\mathfrak{h}]$ in degree one. The PBW-theorem says that the associated graded algebra is the skew group ring $(W \ltimes \mathbb{C}[\mathfrak{h}]) \widehat{\otimes} \widehat{\mathbb{C}[\mathfrak{h}^*]}_{\mathfrak{b}}$ and the claim of the lemma on the level of associated graded spaces is clear. Since multiplication is a filtration preserving map it follows that it is also an isomorphism.

The second claim now follows from the fact that $\widehat{\mathbb{C}[\mathfrak{h}^*]}_{p_j} e_j w = w \widehat{\mathbb{C}[\mathfrak{h}^*]}_{p_k} e_k$ for all $w \in W_{j,k}$ and $W_{i,j} \cdot W_{j,k} = W_{i,k}$. \square

Lemma 3.3.4. *In $\widehat{H}_c(W, \mathfrak{h})_b$ we have $[x, e_i]e_i = 0$, for all $x \in \mathfrak{h}^* \subset \mathbb{C}[\mathfrak{h}]$, $1 \leq i \leq l$.*

Proof. Let $s \in W$ be a reflection. Recall the vectors $\alpha_s \in \mathfrak{h}$ and $\alpha_s^\vee \in \mathfrak{h}^*$ as defined in (2.1). It is shown in section 3.5 of [18] that the functional α_s^\vee can be extend to a \mathbb{C} -linear operator on $\mathbb{C}[\mathfrak{h}^*]$ by setting

$$\alpha_s^\vee(ff') = \alpha_s^\vee(f)f' + f\alpha_s^\vee(f') - \lambda_s \frac{\alpha_s^\vee(f)\alpha_s^\vee(f')}{\alpha_s^\vee(\alpha_s)}\alpha_s, \quad \forall f, f' \in \mathbb{C}[\mathfrak{h}^*]$$

This operator satisfies

$$[x, f] = - \sum_{s \in S} \mathbf{c}(s) \frac{x(\alpha_s)\alpha_s^\vee(f)}{\alpha_s^\vee(\alpha_s)} s \quad \forall x \in \mathfrak{h}^*. \quad (3.3.5)$$

It is also shown that $\alpha_s^\vee(f) = 0$ for all $s \in S$ and $f \in \mathbb{C}[\mathfrak{h}^*]^W$. Therefore α_s^\vee extends to an operator on $\widehat{\mathbb{C}[\mathfrak{h}^*]}_{\mathbf{p}}$ such that relation (3.3.5) holds for $f \in \widehat{\mathbb{C}[\mathfrak{h}^*]}_{\mathbf{p}}$. Applying α_s^\vee to e_i and using the fact that e_i is an idempotent gives

$$\alpha_s^\vee(e_i) \left(1 - 2e_i + \lambda_s \frac{\alpha_s^\vee(e_i)}{\alpha_s^\vee(\alpha_s)} \alpha_s \right) = 0.$$

Multiplying by e_i and using the fact that Lemma 3.3.2 implies that $\widehat{\mathbb{C}[\mathfrak{h}^*]}_{\mathbf{p}} e_i \simeq \widehat{\mathbb{C}[\mathfrak{h}^*]}_{p_i}$, which is a domain, we must have

$$e_i \alpha_s^\vee(e_i) = 0 \quad \text{or} \quad e_i \alpha_s^\vee(e_i) = \frac{\lambda_s}{\alpha_s^\vee(\alpha_s)} \alpha_s^{-1} e_i.$$

However, α_s is invertible in the local ring $\widehat{\mathbb{C}[\mathfrak{h}^*]}_{p_i}$ if and only if $\alpha_s(p_i) \neq 0$ if and only if $s \cdot e_i \neq e_i$. Therefore, multiplying the expression for $[x, e_i]$ given in (3.3.5) on the right by e_i gives zero. \square

Proposition 3.3.6. *For $i = 1, \dots, l$:*

(1) *we have an isomorphism of completed algebras*

$$\theta_i : \widehat{H}_{c'}(W_{p_i}, \mathfrak{h})_{p_i} \xrightarrow{\sim} e_i \widehat{H}_c(W, \mathfrak{h})_b e_i,$$

(2) *the functor $e_i : \widehat{H}_c(W, \mathfrak{h})_b\text{-mod} \rightarrow \widehat{H}_{c'}(W_{p_i}, \mathfrak{h})_{p_i}\text{-mod}$, $M \mapsto e_i M$ is an equivalence of categories,*

(3) *we have an isomorphism of commutative algebras*

$$\phi_i : Z(\widehat{H}_c(W, \mathfrak{h})_b) \xrightarrow{\sim} Z(\widehat{H}_{c'}(W_{p_i}, \mathfrak{h})_{p_i}).$$

Proof. By Lemma 3.3.3 we can define a map

$$\theta_i : \widehat{H}_{c'}(W_{p_i}, \mathfrak{h})_{p_i} = (\mathbb{C}W_{i,i} \otimes \mathbb{C}[\mathfrak{h}]) \widehat{\otimes} \widehat{\mathbb{C}[\mathfrak{h}^*]}_{p_i} \rightarrow e_i \widehat{H}_c(W, \mathfrak{h})_b e_i,$$

by $f \mapsto e_i f e_i$, which is an isomorphism of topological vector spaces. Note that e_i commutes with elements from $\widehat{\mathbb{C}[\mathfrak{h}^*]}_{p_i}$ and $W_{p_i} = W_{i,i}$. To show that this is an algebra morphism we must show that

$$e_i x_1 x_2 e_i = e_i x_1 e_i x_2 e_i, \quad \forall x_1, x_2 \in \mathfrak{h}^* \subset \mathbb{C}[\mathfrak{h}]$$

and

$$e_i [y, x]_1 e_i = [e_i y e_i, e_i x e_i]_2, \quad \forall y \in \mathfrak{h}, x \in \mathfrak{h}^*,$$

where $[-, -]_1$ denotes the commutator in $\widehat{H}_{c'}(W_{p_i}, \mathfrak{h})_{p_i}$ and $[-, -]_2$ the commutator in $\widehat{H}_c(W, \mathfrak{h})_b$. The first equality follows from Lemma 3.3.4 and the second follows directly from the relations (2.1.1), noting that e_i commutes with y .

The decomposition (3.3.1) of $\widehat{H}_c(W, \mathfrak{h})_b$ allows us to think of $\widehat{H}_c(W, \mathfrak{h})_b$ as a “matrix algebra”. In particular, the centre of $\widehat{H}_c(W, \mathfrak{h})_b$ is contained in $\bigoplus_{i=1}^l e_i \widehat{H}_c(W, \mathfrak{h})_b e_i$ and the projection map $u_i : \widehat{H}_c(W, \mathfrak{h})_b \rightarrow e_i \widehat{H}_c(W, \mathfrak{h})_b e_i$ induces an isomorphism $u_i : Z(\widehat{H}_c(W, \mathfrak{h})_b) \rightarrow Z(e_i \widehat{H}_c(W, \mathfrak{h})_b e_i)$. Therefore, we have

$$\phi_i = \theta_i^{-1} \circ u_i : Z(\widehat{H}_c(W, \mathfrak{h})_b) \xrightarrow{\sim} Z(\widehat{H}_{c'}(W_{p_i}, \mathfrak{h})_{p_i}).$$

□

Corollary 3.3.7. *Let $p \in \mathfrak{h}^*$ and $b \in \mathfrak{h}^*/W$ as above. Let b_i be the image of p_i in \mathfrak{h}^*/W_{p_i} . Then, we have an isomorphism of schemes*

$$\pi_W^{-1}(a) \simeq \pi_{W_{p_i}}^{-1}(b_i).$$

Proof. Since the isomorphism θ_i of Proposition 3.3.6 maps the space \mathfrak{n}_{p_i} onto $e_i \mathfrak{m}_b e_i$, the map ϕ_i of Proposition 3.3.6 satisfies

$$\phi_i(\mathfrak{m}_b \cdot Z(\widehat{H}_c(W, \mathfrak{h})_b)) = \mathfrak{n}_{p_i} \cdot Z(\widehat{H}_c(W_{p_i}, \mathfrak{h})_{p_i}).$$

It is proved in [2, Lemma 3.9] that $Z(\widehat{H}_c(W, \mathfrak{h})_b)$ is the completion of $Z(H_c(W, \mathfrak{h}))$ with respect to the ideal generated by \mathfrak{m}_b and, similarly, that $Z(\widehat{H}_{c'}(W_{p_i}, \mathfrak{h})_{p_i})$ is the completion of $Z(H_{c'}(W_{p_i}, \mathfrak{h}))$ with respect to the ideal generated by \mathfrak{n}_{p_i} . Therefore, ϕ_i induces an isomorphism of commutative algebras

$$\frac{Z(H_c(W, \mathfrak{h}))}{\mathfrak{m}_b \cdot Z(H_c(W, \mathfrak{h}))} \xrightarrow{\sim} \frac{Z(H_{c'}(W_{p_i}, \mathfrak{h}))}{\mathfrak{n}_{p_i} \cdot Z(H_{c'}(W_{p_i}, \mathfrak{h}))}.$$

□

We denote by $\widehat{H}_{c'}(\mathfrak{h}, W_p)_0$ the completion of $H_{c'}(\mathfrak{h}, W_p)$ with respect to the ideal generated by the augmentation ideal $\mathbb{C}[\mathfrak{h}^*]_+^{W_p}$ of $\mathbb{C}[\mathfrak{h}^*]^{W_p}$. The map $x \mapsto x$, $w \mapsto w$ and $y \mapsto y + y(p)$ for all $x \in \mathfrak{h}^* \subset \mathbb{C}[\mathfrak{h}]$, $w \in W_p$ and $y \in \mathfrak{h} \subset \widehat{\mathbb{C}[\mathfrak{h}^*]_p}$ defines an isomorphism $\widehat{H}_{c'}(W_p, \mathfrak{h})_p \xrightarrow{\sim} \widehat{H}_{c'}(W_p, \mathfrak{h})_0$. Therefore, we will think of the functor $e_1 : \widehat{H}_c(W, \mathfrak{h})_b\text{-mod} \rightarrow \widehat{H}_{c'}(W_p, \mathfrak{h})_p\text{-mod}$ as an equivalence

$$\Phi : \widehat{H}_c(W, \mathfrak{h})_b\text{-mod} \xrightarrow{\sim} \widehat{H}_{c'}(W_p, \mathfrak{h})_0\text{-mod}.$$

Now consider the generalized Verma module $\Delta(p, \lambda)$. Since $\mathfrak{m}_b \cdot (1 \otimes \lambda) = 0$ and \mathfrak{m}_b is central, $\Delta(p, \lambda)$ is a $\widehat{H}_c(\mathfrak{h}, W)_b$ -module.

Lemma 3.3.8. *Let $p \in \mathfrak{h}^*$ and $\lambda \in \text{Irr}(W_p)$, then $\Phi(\Delta(p, \lambda)) \simeq \Delta(\lambda)$.*

Proof. As a $\widehat{H}_c(W, \mathfrak{h})_b$ -module,

$$\Delta(p, \lambda) = \widehat{H}_c(W, \mathfrak{h})_b \otimes_{\widehat{\mathbb{C}[\mathfrak{h}^*]_b} \rtimes W_p} \lambda = \bigoplus_{i,j=1}^l e_i \widehat{H}_c(W, \mathfrak{h})_b e_j \otimes_{\widehat{\mathbb{C}[\mathfrak{h}^*]_b} \rtimes W_p} \lambda.$$

Recall that $\widehat{\mathbb{C}[\mathfrak{h}^*]_b}$ acts on λ by evaluation at p . Therefore, $e_i \cdot \lambda = 0$ for all $i \neq 1$ and

$$\widehat{\mathbb{C}[\mathfrak{h}^*]_b} \rtimes W_p = \widehat{\mathbb{C}[\mathfrak{h}^*]_b} e_1 \rtimes W_p \oplus \left(\bigoplus_{i=2}^l \widehat{\mathbb{C}[\mathfrak{h}^*]_b} e_i \right) \rtimes W_p$$

implies that

$$\Delta(p, \lambda) = \widehat{H}_c(W, \mathfrak{h})_b \otimes_{\widehat{\mathbb{C}[\mathfrak{h}^*]_b} \rtimes W_p} \lambda = \bigoplus_{i=1}^l e_i \widehat{H}_c(W, \mathfrak{h})_b e_1 \otimes_{\widehat{\mathbb{C}[\mathfrak{h}^*]_b} e_1 \rtimes W_p} \lambda.$$

Thus,

$$e_1 \cdot \Delta(p, \lambda) = e_1 \widehat{H}_c(W, \mathfrak{h})_b e_1 \otimes_{\widehat{\mathbb{C}[\mathfrak{h}^*]_b} e_1 \rtimes W_p} \lambda.$$

This implies that $\Phi(\Delta(p, \lambda)) \simeq \Delta(\lambda)$. □

Theorem 3.3.9. *Assume that the fixed point x_λ is contained in the smooth locus of $X_{c'}(W_p)$. Then,*

- (1) *The center of $H_c(\mathfrak{h}, W)$ surjects onto $E_{(p, \lambda)} := \text{End}_{H_c(\mathfrak{h}, W)}(\Delta(p, \lambda))$.*
- (2) *The algebra $E_{(p, \lambda)}$ is a polynomial ring of dimension $\dim \mathfrak{h}$.*
- (3) *$\Delta(p, \lambda)$ is a cyclic $E_{(p, \lambda)}$ -module.*

Proof. Since we can identify

$$\text{End}_{H_c(\mathfrak{h}, W)}(\Delta(p, \lambda)) = \text{End}_{\widehat{H}_c(\mathfrak{h}, W)}(\Delta(p, \lambda)),$$

the theorem follows from Theorem 3.1.3, Proposition 3.3.6 (2) and Lemma 3.3.8. □

Let $p \in \mathfrak{h}^*$, $\lambda \in \text{Irr} W_p$ and $w \in W$. Then, clearly $\Delta(p, \lambda) \simeq \Delta(w(p), w(\lambda))$, where $w(\lambda)$ is the representation of $W_{w(p)}$ corresponding to λ under the isomorphism $w : W_p \xrightarrow{\sim} W_{w(p)}$ of conjugation. Therefore, if \mathbf{b} is the image of p in \mathfrak{h}^*/W , then we denote by $\Omega_{\mathbf{b}, \lambda}$ the support of the $Z_c(W)$ -module $\Delta(p, \lambda)$, thought of as a subscheme of $X_c(W)$. Then, Theorem 3.3.9 says that if $x_\lambda \in X_{c'}(W_p)$ is contained in the smooth locus then $\Omega_{\mathbf{b}, \lambda}$ is isomorphic to \mathbb{A}^n , as a closed subscheme of $X_c(W)$. For $\mathbf{a} \in \mathfrak{h}/W$, let $\Omega_{\mathbf{b}, \lambda, \mathbf{a}}$ denote the scheme theoretic intersection $\Omega_{\mathbf{b}, \lambda} \cap \varpi^{-1}(\mathbf{a})$. If $x_\lambda \in X_{c'}(W_p)$ is contained in the smooth locus then the fact that $\Delta(p, \lambda)$ is a free $\mathbb{C}[\mathfrak{h}]^W$ -module, together with Theorem 3.3.9, implies that

$$\text{Supp } \Delta(p, \lambda, \mathbf{a}) = \Omega_{\mathbf{b}, \lambda, \mathbf{a}}.$$

In this case, we have $\dim \mathbb{C}[\Omega_{\mathbf{b}, \lambda, \mathbf{a}}] = \dim \lambda$.

We could have worked instead with $q \in \mathfrak{h}$, \mathbf{a} the image of q in \mathfrak{h}/W , $\mu \in W_q$ and $\mathbf{b} \in \mathfrak{h}^*/W$. We set $\mathcal{U}_{\mathbf{a}, \mu} = \text{Supp} \nabla(q, \mu)$ and $\mathcal{U}_{\mathbf{a}, \mu, \mathbf{b}} = \text{Supp} \nabla(q, \mu, \mathbf{b})$. The obvious analogue of Theorem 3.3.9 holds in this situation.

3.4. An equivalence of categories. Let $H_c(W)\text{-mod}_{\mathbf{b}, \lambda}$ denote the category of finitely generated $H_c(W)$ -modules scheme-theoretically supported on $\Omega_{\mathbf{b}, \lambda}$ i.e. those modules M such that $I \cdot M = 0$, where I is the ideal defining $\Omega_{\mathbf{b}, \lambda}$. The category of coherent $\mathcal{O}_{\Omega_{\mathbf{b}, \lambda}}$ -modules is denoted $\text{Coh}(\Omega_{\mathbf{b}, \lambda})$. In this section we prove the following theorem:

Theorem 3.4.1. *If $x_\lambda \in X_{c'}(W_p)$ is contained in the smooth locus, then the functor*

$$F : H_c(W)\text{-mod}_{\mathbf{b}, \lambda} \rightarrow \text{Coh}(\Omega_{\mathbf{b}, \lambda}), \quad F(M) = \widetilde{\text{Hom}_{H_c(W)}(\Delta(p, \lambda), M)},$$

is an equivalence of categories with quasi-inverse $N \mapsto G(N) := \Delta(p, \lambda) \otimes_{Z_c(W)} \Gamma(\Omega_{\mathbf{b}, \lambda}, N)$.

Lemma 3.4.2. *Let $e : H_c(W)\text{-mod} \rightarrow \text{Coh}(X_c(W))$ be the functor $M \mapsto eM$.*

- (1) *The functor e is an equivalence if and only if $X_c(W)$ is smooth.*
- (2) *If $x_\lambda \in X_{c'}(W_p)$ is contained in the smooth locus, then e defines an equivalence*

$$e : H_c(W)\text{-mod}_{\mathbf{b}, \lambda} \xrightarrow{\sim} \text{Coh}(\Omega_{\mathbf{b}, \lambda}).$$

Proof. Part 1: this is well-known but we sketch a proof for the readers' convenience. Via the Satake isomorphism, [12, Theorem 3.1], we can rephrase the statement as: the $(H_c(W), eH_c(W)e)$ -bimodule $H_c(W)e$ induces an equivalence $H_c(W)\text{-mod} \rightarrow eH_c(W)e\text{-mod}$ if and only if $Z_c(W)$ is a regular algebra. The double centralizer theorem [12, Theorem 1.5] implies that it is sufficient to show that $H_c(W)e$ is a progenerator of $H_c(W)\text{-mod}$. Since $H_c(W)e$ is a summand of $H_c(W)$, it is projective. It will generate the category $H_c(W)\text{-mod}$ if and only if $eM \neq 0$ for all $M \in H_c(W)\text{-mod}$ if and only if $eM \neq 0$ for all simple $H_c(W)$ -modules. The fact that $Z_c(W)$ is a regular algebra implies that $eM \neq 0$ for all simple $H_c(W)$ -modules follows from [12, Theorem 1.7]. Conversely, if there exists a simple module M such that $eM = 0$ then M is not isomorphic to the regular

representation as a W -module. Hence, [12, Theorem 1.7] again implies that the support of M is contained in the singular locus of $X_c(W)$.

Part 2: by Lemma 3.1.1, the assumption of part (2) implies that $\Omega_{b,\lambda} \subseteq X_c(W)_{\text{sm}}$. Since $\Omega_{b,\lambda} \cap X_c(W)_{\text{sing}} = \emptyset$, Hilbert's Nullstellensatz implies that $I(\Omega_{b,\lambda}) + I(X_c(W)_{\text{sing}}) = Z_c(W)$ and we can find a characteristic function $f \in Z_c(W)$ taking the value 1 at all points of $\Omega_{b,\lambda}$ and vanishing on $X_c(W)_{\text{sing}}$. Replacing $H_c(W)$ by its localization at f and $Z_c(W)$ by its localization, we may assume that $Z_c(W)$ is a regular affine algebra and $H_c(W)$ an Azumaya algebra over $Z_c(W)$. Notice also that the proof of part (1) still holds after localization. Recall that the centre $Z(\mathcal{A})$ of an abelian category \mathcal{A} is defined to be the ring of endomorphisms $\text{End}_{\mathcal{A}}(\text{id}_{\mathcal{A}})$ of the identity functor. For a Noetherian k -algebra A , the centre of $A\text{-mod}$ is canonically isomorphic to the centre of A . The equivalence e induces an isomorphism $Z(H_c(W)\text{-mod}) \xrightarrow{\sim} Z(\text{Coh}(X_c(W))) \xrightarrow{\sim} Z(H_c(W))$

$$Z(H_c(W)) \xrightarrow{\sim} Z(H_c(W)\text{-mod}) \xrightarrow{\sim} Z(\text{Coh}(X_c(W))) \xrightarrow{\sim} Z(H_c(W))$$

is just the identity map. Let $I = I(\Omega_{b,\lambda})$. We can identify $H_c(W)\text{-mod}_{b,\lambda} = \{M \in H_c(W)\text{-mod} \mid i_M = 0 \ \forall i \in I\}$, where $i_M \in \text{End}_{H_c(W)}(M)$ is the endomorphism defined by $i \in Z(H_c(W)\text{-mod})$. Similarly, $\text{Coh}(\Omega_{b,\lambda}) = \{\mathcal{F} \in \text{Coh}(X_c(W)) \mid i_{\mathcal{F}} = 0 \ \forall i \in I\}$. From this it follows that $e : H_c(W)\text{-mod}_{b,\lambda} \xrightarrow{\sim} \text{Coh}(\Omega_{b,\lambda})$. \square

Then, Theorem 3.4.1 follows from

Lemma 3.4.3. *The Verma module $\Delta(p, \lambda)$ is a pro-generator in $H_c(W)\text{-mod}_{b,\lambda}$.*

Proof. By Lemma 3.4.2 it suffices to show that $e\widetilde{\Delta(p, \lambda)}$ is a projective generators of $\text{Coh}(\Omega_{b,\lambda})$. But, by Theorem 3.3.9, $e\Delta(p, \lambda)$ is the regular representation as a $\mathbb{C}[\Omega_{b,\lambda}]$ -module. Therefore $e\widetilde{\Delta(p, \lambda)} \simeq \mathcal{O}_{\Omega_{b,\lambda}}$ as sheaves on $\Omega_{b,\lambda}$. \square

Remark 3.4.4. Note that $\Delta(p, \lambda)$ is not projective as an object in $H_c(W)\text{-mod}$ because the Gelfand-Kirillov dimension of $\Delta(p, \lambda)$ is $\dim \mathfrak{h}$, whereas the Gelfand-Kirillov dimension of $H_c(W)$ is $2 \dim \mathfrak{h}$. If I is the ideal of $Z_c(W)$ defining $\Omega_{b,\lambda}$, then set $H_c(W)_{p,\lambda} := H_c(W)/\langle I \rangle$. One can reinterpret Lemma 3.4.3 as saying that $H_c(W)_{p,\lambda}$ is a *split* Azumaya algebra over $\Omega_{b,\lambda}$, with splitting bundle $\Delta(p, \lambda)$.

3.5. Lagrangian subvarieties. It is shown in [19, Proposition 4.5] that $X_c(W)$ is a symplectic variety, see [14] for the definition and properties of symplectic varieties. This implies that the smooth locus $X_c(W)_{\text{sm}}$ is a symplectic leaf in $X_c(W)$ and hence has codimension at least two.

Definition 3.5.1. A reduced subvariety Y of $X_c(W)$ is said to a Lagrangian subvariety if $Y_{\text{sm}}^i \cap X_c(W)_{\text{sm}}$ is a *non-empty* Lagrangian submanifold of $X_c(W)_{\text{sm}}$ for each irreducible component Y^i of Y .

The goal of this subsection is to prove the following proposition. It is a consequence of Gabber's Integrability Theorem, [16].

Proposition 3.5.2. *Let $b \in \mathfrak{h}^*/W$. Then $\pi^{-1}(b)_{\text{red}}$ is a Lagrangian subvariety of $X_c(W)$.*

Let \mathfrak{m}_b be the maximal ideal in $\mathbb{C}[\mathfrak{h}^*]^W$ corresponding to b . First, we show

Lemma 3.5.3. *Let J be the radical of the ideal generated by \mathfrak{m}_b in $Z_c(W)$. Then, J is involutive i.e. $\{J, J\} \subseteq J$.*

Proof. It is clear from the definition of the Poisson bracket on $Z_c(W)$ that the ideal generated by \mathfrak{m}_b in $Z_c(W)$ is involutive. However, it seems that this does not in general imply that J is involutive. Therefore, we need to work a bit harder. Since we have not assumed any smoothness condition

on $X_c(W)$, we are also unable to use results from previous sections. Let $p_1, \dots, p_k \in \mathfrak{h}^*$ be the elements in the orbit \mathbf{b} . Set

$$M = \bigoplus_{i=1}^k \left(\bigoplus_{\lambda \in W_{p_i}} \Delta(p_i, \lambda) \right).$$

Let $Y = (\text{Supp } M)_{\text{red}}$. Then, I claim that $Y = V(J)$. Since $\mathfrak{m}_{\mathbf{b}} \cdot M = 0$, we have $Y \subset V(J)$. Let $x \in V(J)$ and choose some simple $H_c(W)$ -module L supported on x . As a $\mathbb{C}[\mathfrak{h}^*]$ -module, $L = \bigoplus_{i=1}^k L_{p_i}$, where L_{p_i} is supported at p_i . Without loss of generality, we may assume that $L_{p_1} \neq 0$. Let $\lambda \in L_{p_1}$ be an irreducible W_{p_1} -module in the socle of L_{p_1} . Then, there is a non-zero homomorphism $\Delta(p_1, \lambda) \rightarrow L$. This implies that $\text{Hom}_{H_c(W)}(M, L) \neq 0$ and hence $x \in Y$.

Now, let $\mathbb{C}[\epsilon]$ be the dual numbers, so that $\epsilon^2 = 0$. It will be easier to work, via the Satake isomorphism, with the spherical subalgebra $eH_c(W)e$. The usual rational Cherednik algebra $H_{t,c}(W)$ has an additional parameter t , which we have assume throughout is set to zero. Specializing instead to $t = \epsilon$, we have a $\mathbb{C}[\epsilon]$ -algebra $eH_{\epsilon,c}(W)e$ such that $eH_{\epsilon,c}(W)e/\epsilon eH_{\epsilon,c}(W)e \simeq eH_c(W)e$. Then the Poisson structure on $eH_c(W)e$ is constructed as in [16]. We can define $\Delta_{\epsilon}(p, \lambda)$ in the obvious way. It is a $H_{\epsilon,c}(W)$ -module, free over $\mathbb{C}[\epsilon]$. This gives us a $eH_{\epsilon,c}(W)e$ -module eM_{ϵ} , free over $\mathbb{C}[\epsilon]$. This freeness implies that condition (1.2) of [16] is satisfied. Then, the fact that J is involutive is a consequence of [16, Theorem II], together with the fact that the Satake isomorphism is an isomorphism of Poisson algebras. \square

Proof of Proposition 3.5.2. Let $n = \dim \mathfrak{h}$. The maximal ideal $\mathfrak{m}_{\mathbf{b}}$ in $\mathbb{C}[\mathfrak{h}]^W$ is generated by a regular sequence f_1, \dots, f_n . By Lemma 2.2.1 (1), they also form a regular sequence in $Z_c(W)$. Therefore, the fact that the morphism π is flat, together with [21, Corollary 9.6 (ii)], implies that each irreducible component of $\pi^{-1}(\mathbf{b})_{\text{red}}$ is n -dimensional. Let Y^1, \dots, Y^k be these irreducible components. Let $\delta := \prod_{s \in S} \alpha_s \in \mathbb{C}[\mathfrak{h}]$; it is a semi-invariant with respect to W . Therefore, there exists some $k \in \mathbb{N}$ such that $\delta^k \in \mathbb{C}[\mathfrak{h}]^W$. The sequence f_1, \dots, f_n extends to a regular sequence $f_1, \dots, f_n, \delta^k$ in $Z_c(W)$. This implies that, for each i , $\dim Y^i \cap V(\delta^k) = n - 1$ if $Y^i \cap V(\delta^k) \neq \emptyset$. In particular, $Y^i \setminus V(\delta^k) \neq \emptyset$ for all i . The Dunkl embedding, as explained in [12, §4], shows that $H_c(W)[\delta^{-k}] \simeq \mathbb{C}[\mathfrak{h} \times \mathfrak{h}_{\text{reg}}^*] \rtimes W$ where $\mathfrak{h}_{\text{reg}}^*$ is the set of points in \mathfrak{h}^* with trivial W -stabilizer. Since the centre of $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}_{\text{reg}}^*] \rtimes W$ is a regular domain, it follows that $Y^i \setminus V(\delta^k) \subset X_c(W)_{\text{sm}}$. Thus, we have shown that $Y^i \cap X_c(W)_{\text{sm}}$ is non-empty for all components Y^i of $\pi^{-1}(\mathbf{a})_{\text{red}}$. Let J^i be the ideal defining Y^i . It is a minimal prime over the ideal J . By Lemma 3.5.3, J is an involutive ideal. Therefore, J^i is an involutive ideal, see [9, Lemma 2.1]. Choose some point $x \in Y^i \cap X_c(W)_{\text{sm}}$. Then, [8, Proposition 1.5.1] says that $T_x Y^i$ is a coisotropic subspace of $T_x X_c(W)$. But $\dim T_x Y = n$, therefore it is actually a Lagrangian subspace as required. \square

Remark 3.5.4. (1) If $X_c(W)$ is a smooth variety, as will be the case for the symmetric group, then it is a symplectic manifold and Proposition 3.5.2 shows that $\pi^{-1}(\mathbf{b})_{\text{red}}$ is the disjoint union of finitely many Lagrangian submanifolds of $X_c(W)$.
(2) An analogous result to Proposition 3.5.2 holds when one considers $\varpi^{-1}(\mathbf{a})$ for some $\mathbf{a} \in \mathfrak{h}/W$.

3.6. Batalin-Vilkoviski structures. Let I be the ideal defining the closed subscheme $\Omega_{\mathbf{b},\lambda}$ in $X_c(W)$. We assume that $\Omega_{\mathbf{b},\lambda}$ is smooth so that $N_{\mathbf{b},\lambda}^{\vee} := I/I^2$ is a locally free $Z_c(W)/I$ -module. It is the module of sections of the conormal bundle of $\Omega_{\mathbf{b},\lambda}$ in $X_c(W)$. Its dual $N_{\mathbf{b},\lambda} := (I/I^2)^{\vee}$ is the module of sections of the normal bundle of $\Omega_{\mathbf{b},\lambda}$ in $X_c(W)$. Let $\lambda^* = \text{Hom}_{\mathbb{C}}(\lambda, \mathbb{C})$, an irreducible right W_p -module. This extends, via evaluation at p , to a right $\mathbb{C}[\mathfrak{h}^*] \rtimes W_p$ -module and we let $\Delta(p, \lambda)^{op}$ be the right $H_c(W)$ -module obtained by inducing from λ^* .

Theorem 3.6.1. Assume that $x_\lambda \in X_c(W_p)$ is contained in the smooth locus $X_c(W_p)_{\text{sm}}$. Then, the algebra $\text{Tor}_\bullet^{\text{Hc}}(\Delta(p, \lambda)^{\text{op}}, \Delta(p, \lambda))$ admits a canonical structure of a Gesterhaber algebra such that

$$\text{Tor}_\bullet^{\text{Hc}}(\Delta(p, \lambda)^{\text{op}}, \Delta(p, \lambda)) \simeq \wedge^\bullet \mathbf{N}_{b, \lambda}^\vee \quad (3.6.2)$$

as Gesterhaber algebras.

Moreover, the sheaf $\text{Ext}_\bullet^{\text{Hc}}(\Delta(p, \lambda), \Delta(p, \lambda))$ admits a canonical structure of Gesterhaber module over the Gesterhaber algebra $\text{Tor}_\bullet^{\text{Hc}}(\Delta(p, \lambda)^{\text{op}}, \Delta(p, \lambda))$ such that

$$\text{Ext}_\bullet^{\text{Hc}}(\Delta(p, \lambda), \Delta(p, \lambda)) \simeq \wedge^\bullet \mathbf{N}_{b, \lambda} \quad (3.6.3)$$

as Gesterhaber modules, compatible in the obvious sense with the identification (3.6.2).

Proof. As in the proof of Lemma 3.4.2, the assumption on x_λ implies that there exists an affine open subset U of $X_c(W)_{\text{sm}}$ such that $\Omega_{b, \lambda} \subset U$. It follows from Proposition 3.5.2 that $\Omega_{b, \lambda}$ is a smooth, Lagrangian subvariety of $X_c(W)$. Let $A = \mathbb{C}[U]$ and $B = \mathbb{C}[\Omega_{b, \lambda}]$. Then, as explained in the appendix, $\wedge^\bullet \mathbf{N}_{b, \lambda}^\vee$ admits the structure of a Gesterhaber algebra and $\wedge^\bullet \mathbf{N}_{b, \lambda}$ is a Gesterhaber module over $\wedge^\bullet \mathbf{N}_{b, \lambda}^\vee$. Theorem 8.0.7 says that we have isomorphisms

$$\text{Tor}_\bullet^A(B, B) \simeq \wedge^\bullet \mathbf{N}_{b, \lambda}^\vee, \quad \text{Ext}_A^\bullet(B, B) \simeq \wedge^\bullet \mathbf{N}_{b, \lambda}$$

of Gesterhaber algebras and Gesterhaber modules respectively. Therefore, the theorem follows from Lemma 3.4.2 which implies that $e \cdot - : \text{Hc}(W)|_{U\text{-mod}} \rightarrow A\text{-mod}$ is an equivalence, sending $\Delta(p, \lambda)$ to B and that the corresponding result for right $\text{Hc}(W)|_U$ -modules also sends $\Delta(p, \lambda)^{\text{op}}$ to B . \square

Remark 3.6.4. The key point, hidden in the proof of Theorem 3.6.1, is that the algebra $\text{Hc}(W)$ admits a canonical algebra deformation, via the parameter t , such that module $\Delta(p, \lambda)$ also deforms naturally to a $\text{H}_{t, c}(W)$ -module.

Recall from Theorem 3.1.3 that the graded character of $\mathbb{C}[\Omega_{0, \lambda}]$ is $q^{-b_\lambda} f_{\lambda^*}(q) \prod_{i=1}^n (1 - q^{d_i})^{-1}$, where d_1, \dots, d_n are the degrees of (W, \mathfrak{h}) . Since $\mathbb{C}[\Omega_{0, \lambda}]$ is a polynomial ring, this implies that there exists² integers $0 < e_1 \leq \dots \leq e_n$ such that

$$q^{-b_\lambda} f_{\lambda^*}(q) \prod_{i=1}^n (1 - q^{e_i})^{-1} = \prod_{i=1}^n (1 - q^{d_i}). \quad (3.6.5)$$

Corollary 3.6.6. We have

$$\text{Tor}_\bullet^{\text{Hc}(W)}(\Delta(\lambda)^{\text{op}}, \Delta(\lambda)) = q^{-b_\lambda} f_{\lambda^*}(q) \prod_{i=1}^n \frac{1 + tq^{-e_i}}{1 - q^{d_i}}, \quad (3.6.7)$$

and

$$\text{Ext}_\bullet^{\text{Hc}(W)}(\Delta(\lambda), \Delta(\lambda)) = q^{-b_\lambda} f_{\lambda^*}(q) \prod_{i=1}^n \frac{1 + tq^{e_i}}{1 - q^{d_i}}. \quad (3.6.8)$$

Proof. We see from the definition of the integers e_i that the graded character of $T_{x_\lambda} \Omega_{0, \lambda}$ is given by $\sum_{i=1}^n q^{-e_i}$. The proof of Corollary 3.1.5 shows that $T_{x_\lambda} X_c(W) = T_{x_\lambda} \Omega_{0, \lambda} \oplus (T_{x_\lambda} \Omega_{0, \lambda})^\perp$ with respect to the symplectic form on $X_c(W)$ and we may identify $(T_{x_\lambda} \Omega_{0, \lambda})^\perp = (T_{\Omega_{p, \lambda}} X_c(W))_{x_\lambda}$ as \mathbb{C}^\times -representations. This implies that

$$\text{ch}_q(T_{\Omega_{p, \lambda}} X_c(W))_{x_\lambda} = \sum_{i=1}^n q^{e_i}.$$

²For an arbitrary representation $\lambda \in \text{Irr}(W)$, it is not possible to find integers $0 < e_1 \leq \dots \leq e_n$ such that (3.6.5) holds. This is related to the fact that x_λ is always in the singular locus of $X_c(W)$, regardless of the parameter c .

Then formula (3.6.8) follows from the fact that

$$\mathrm{Ext}_{\mathrm{Hc}(W)}^\bullet(\Delta(\lambda), \Delta(\lambda)) = \mathbb{C}[\Omega_{0,\lambda}] \otimes \wedge^\bullet(T_{\Omega_{p,\lambda}} \mathbf{X}_{\mathbf{c}}(W))_{x_\lambda}$$

as graded vector spaces, together with the fact that if V is a graded vector space with character $q^{m_1} + \dots + q^{m_k}$ then the bigraded character of $\wedge^\bullet V$ is given by $(1 + tq^{m_1}) \dots (1 + tq^{m_k})$. The proof of formula (3.6.7) is similar. \square

4. WILSON'S CALOGERO-MOSER SPACE

In this section, we recall some of the basic properties of the Wilson's completion of the Calogero-Moser phase space.

4.1. The Calogero-Moser space. The Calogero-Moser space CM_n is a completion of the phase space associated to the Calogero-Moser integrable system, which was introduced by Wilson in the seminal paper [34]. It is a smooth affine variety of dimension $2n$ and a symplectic manifold. Denote by \mathfrak{g} the space of all $n \times n$ matrices over \mathbb{C} and define $\overline{\mathrm{CM}}_n \subset \mathfrak{g} \times \mathfrak{g}$ to be the set of all pairs (X, Y) such that the rank of $[X, Y] + I_n$ equals one, where $I_n \in \mathfrak{g}$ is the identity matrix. The group PGL_n acts on $\overline{\mathrm{CM}}_n$ by simultaneous conjugation, $g \cdot (X, Y) = (\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y))$. It is shown in [34, Corollary 1.5] that this action is free.

Definition 4.1.1. The *Calogero-Moser space* CM_n is defined to be the categorical (= geometric) quotient $\overline{\mathrm{CM}}_n // \mathrm{PGL}_n$.

Wilson also gave a slightly different construction of CM_n which allows one to replace the group PGL_n by GL_n . Let V be the vectorial representation for GL_n and define

$$\widetilde{\mathrm{CM}}_n := \{(X, Z, v, w) \in \mathfrak{g} \oplus \mathfrak{g} \oplus V \oplus V^* \mid [X, Z] - v \circ w = -I_n\}.$$

The group GL_n acts on $\mathfrak{g} \times V$ by $g \cdot (X, v) = (\mathrm{Ad}_g(X), g \cdot v)$. Via the trace form, we identify \mathfrak{g}^* with \mathfrak{g} and $T^*(\mathfrak{g} \times V)$ with $\mathfrak{g} \times \mathfrak{g} \times V \times V^*$. The induced GL_n -action on $T^*(\mathfrak{g} \times V)$, is Hamiltonian with moment map

$$\mu : T^*(\mathfrak{g} \times V) \rightarrow \mathfrak{g}, \quad (X, Y; v, w) \mapsto [X, Y] - v \circ w.$$

Then $\widetilde{\mathrm{CM}}_n = \mu^{-1}(-I_n)$. It is shown in [34, Corollary 1.5] that GL_n acts freely on $\widetilde{\mathrm{CM}}_n$ and Proposition 1.7 of *loc. cit.* says that the differential $d\mu$ of μ is surjective at all points in $\widetilde{\mathrm{CM}}_n$. Therefore, the categorical quotient $\mu^{-1}(-I_n) // \mathrm{GL}_n$ is a smooth affine variety. Since it is the Hamiltonian reduction of $T^*(\mathfrak{g} \times V)$, the space $\mu^{-1}(-I_n) // \mathrm{GL}_n$ is naturally a symplectic manifold.

As noted in [34, §1], the projection $\widetilde{\mathrm{CM}}_n \rightarrow \overline{\mathrm{CM}}_n$ induces an isomorphism $\mu^{-1}(-I_n) // \mathrm{GL}_n \simeq \mathrm{CM}_n$.

4.2. \mathbb{C}^\times -fixed points in CM_n . There is a \mathbb{C}^\times -action on CM_n , defined by $\alpha \cdot (X, Y) = (\alpha^{-1}X, \alpha Y)$. In terms of the space $\widetilde{\mathrm{CM}}_n$, the action is

$$\alpha \cdot (X, Y; v, w) = (\alpha^{-1}X, \alpha Y; \alpha^{-1}v, \alpha w).$$

The fixed points of this \mathbb{C}^\times -action were classified in [34, §6] and explicit representatives $(X, Y; v, w)$ of each fixed point given in [34, Lemma 6.9]. We recall this description here.

Each fixed point of CM_n is naturally labeled by a partition λ of n and hence the set of all fixed points of CM_n is in bijection with the set of all partitions of n . Therefore, for each $\lambda \vdash n$, we need to construct a point $\mathbf{X}_\lambda \in \mathrm{CM}_n$. Firstly, we rewrite our partition $\lambda = (\lambda_1, \dots, \lambda_k)$ in Frobenius form. This means that λ is written as the union of hook partitions $(n - r + 1, 1^{r-1})$ of decreasing size such that when we stack one above the other, the largest at the bottom and smallest at the top, we recover the Young diagram of λ . An example is given in figure 4.2.

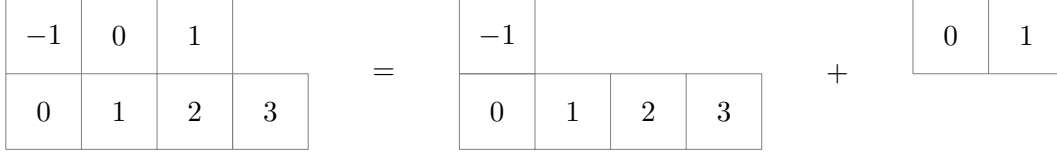


FIGURE 1. The partition $(4, 3)$ in Frobenius form. Here $\{(n_1, r_1), (n_2, r_2)\}$ equals $\{(5, 2), (2, 1)\}$.

Combinatorially, λ is written as an l -tuple of pairs $(n_1, r_1), \dots, (n_l, r_l)$ subject to the restrictions $r_i > r_j$ and $n_i - r_i > n_j - r_j$ if $i < j$. Here $\sum_i n_i = n$ and $1 \leq r_i \leq n_i$ for all i . Given such a pair, we have

$$\mathbf{X}_\lambda = (X, Y; v, w) = (\oplus_{i,j} X_{i,j}, \oplus_i Y_i; v_i, w_j)_{i,j=1 \dots l}$$

where Y_i is the upper-triangular Jordan block of size $n_i \times n_i$ with eigenvalues 0. The matrix $X_{i,i}$ has all diagonals zero except the -1 diagonal (i.e. just below the main diagonal) where the entries from top left to bottom right reads

$$1, 2, \dots, r_i - 1; -(n_i - r_i), \dots, -2, -1. \quad (4.2.1)$$

For $i \neq j$, $X_{i,j}$ is a $n_i \times n_j$ matrix with non-zero entries only on the $r_j - r_i - 1$ diagonal. If $i > j$ then the non-zero diagonal of $X_{i,j}$ has r_i entries equal to n_i followed by $n_i - r_i$ entries equal to zero. If $i < j$, the non-zero diagonal of $X_{i,j}$ has $r_j - 1$ entries equal to 0 followed by $n_j - r_j + 1$ entries equal to $-n_i$. Once (X, Y) have been given in this way, v, w are uniquely defined by the equation $[X, Y] + I_n = v \circ w$.

4.3. Consider the map $\rho : \overline{\mathbf{CM}}_n \rightarrow \mathfrak{g}$ given by $(X, Y) \mapsto Z := YX$. This induces a map $\rho : \mathbf{CM}_n \rightarrow \mathfrak{g} // \mathrm{GL}_n \simeq \mathbb{C}^n / \mathfrak{S}_n$. In order to clearly state our results about the \mathbb{C}^\times -fixed points in \mathbf{CM}_n , we will consider points in $\mathbb{C}^n / \mathfrak{S}_n$ to be elements in the ring $\mathbb{Z}[q^\kappa \mid \kappa \in \mathbb{C}]$, the group ring of the additive group $(\mathbb{C}, +)$. Thus,

$$\mathbb{C}^n / \mathfrak{S}_n \ni \sum_{i=1}^k n_i \kappa_i \quad \leftrightarrow \quad \sum_{i=1}^k n_i q^{\kappa_i} \in \mathbb{Z}[q^\kappa \mid \kappa \in \mathbb{C}].$$

It will become apparent in section 5.4 that this morphism is dominant.

The Young diagram of a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ is the diagram $Y(\lambda) := \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq j \leq \lambda_i, 1 \leq i \leq \lambda_j\} \subset \mathbb{Z}^2$. The *content* of the box (i, j) is $\mathrm{cont}(i, j) := i - j$. We define the residue of λ to be the Laurent polynomial $\mathrm{Res}_\lambda(q) = \sum_{(i,j) \in Y(\lambda)} q^{\mathrm{cont}(i,j)}$. It defines a point in $\mathbb{C}^n / \mathfrak{S}_n$.

We wish to calculate the image of the fixed points \mathbf{X}_λ under ρ . Write $Z = YX = \oplus_{i,j} Z_{i,j}$, where $Z_{i,j}$ is a matrix of size $n_i \times n_j$. Then $Z_{i,j}$ has non-zero entries only on the $r_j - r_i$ diagonal. The square matrix $Z_{i,i}$ has entries only on the main diagonal, from top left to bottom right they are

$$1, 2, \dots, r_i - 1; -(n_i - r_i), \dots, -2, -1, 0. \quad (4.3.1)$$

For $i > j$, the non-zero diagonal of $Z_{i,j}$ has $r_i - 1$ entries equal to n_i followed by $n_i - r_i$ entries equal to 0. If $i < j$ then the non-zero diagonal of $Z_{i,j}$ has $r_j - 1$ entries equal to 0 followed by $n_j - r_j + 1$ entries equal to $-n_i$.

Lemma 4.3.2. *After row reduction Z can be put in the form $\tilde{Z} = \oplus_{i,j} \tilde{Z}_{i,j}$ where $\tilde{Z}_{i,i} = Z_{i,i}$ for all i and $\tilde{Z}_{i,j} = 0$ for $i > j$.*

Proof. The proof is a direct calculation. If the reader really wants to understand the proof we recommend they draw a picture to see what's going on.

Inductively on i , we claim that we can remove the non-zero entries in each row of $Z_{i,j}$, where $i > j$, by taking away some multiple of a certain row above the rows of $Z_{i,j}$ in such a way that all

other blocks remain unchanged. So let us fix $i > j$ and we assume by induction that $Z_{i',j'} = 0$ for all $i' < i$ and $i' > j'$. Write $Z_{i,j} = (z_{\alpha,\beta})_{\alpha,\beta}$, where $1 \leq \alpha \leq n_i$, $1 \leq \beta \leq n_j$. Then, from the above description of Z we see that the only non-zero entries $z_{\alpha,\beta}$ of $Z_{i,j}$ are $z_{\alpha,\alpha+r_j-r_i}$ for $\alpha = 1, \dots, r_i - 1$ (recall that $i > j$ implies that $r_j - r_i > 0$). Now consider the column of Z containing $z_{\alpha,\alpha+r_j-r_i}$. This column intersects the main diagonal of Z in the block $Z_{j,j} = (\hat{z}_{a,b})_{a,b}$ and the diagonal entry of $Z_{j,j}$ in this column is $\hat{z}_{\alpha+r_j-r_i,\alpha+r_j-r_i}$. Since $\alpha \leq r_i - 1$, we have $\alpha + r_j - r_i \leq r_j - 1 < n_j$. Therefore, (4.3.1) implies that $\hat{z}_{\alpha+r_j-r_i,\alpha+r_j-r_i} \neq 0$ and we can certainly take away from the row of Z containing $z_{\alpha,\alpha+r_j-r_i}$ a multiple of the row of Z containing $\hat{z}_{\alpha+r_j-r_i,\alpha+r_j-r_i}$ such that the new value of $z_{\alpha,\alpha+r_j-r_i}$ is zero.

I claim that $\hat{z}_{\alpha+r_j-r_i,\alpha+r_j-r_i}$ is the only non-zero entry of the $(\alpha + r_j - r_i)$ th row. If this is the case, then it is clear that none of the other blocks of Z are changed under this row operation. The induction hypothesis implies that all entries to the left of $\hat{z}_{\alpha+r_j-r_i,\alpha+r_j-r_i}$ are zero. Since $Z_{j,j}$ is diagonal, all the entries to the right of $\hat{z}_{\alpha+r_j-r_i,\alpha+r_j-r_i}$ in $Z_{j,j}$ are also zero. Therefore, any non-zero entry of the row would lie in a block $Z_{j,k}$ with $k > j$. We have $r_k - r_j < 0$. Let $Z_{j,k} = (\bar{z}_{u,v})_{u,v}$. Then, the only non-zero entries of $Z_{j,k}$ are \bar{z}_{u,r_k-r_j+u} for $u = r_j, \dots, n_j$. But the row of Z containing $\hat{z}_{\alpha+r_j-r_i,\alpha+r_j-r_i}$ intersects $Z_{j,k}$ in $(\bar{z}_{\alpha+r_j-r_i,1}, \dots, \bar{z}_{\alpha+r_j-r_i,n_k})$. Now $1 \leq \alpha \leq r_i - 1$ so $\alpha + r_j - r_i < r_j$ which implies $\bar{z}_{\alpha+r_j-r_i,v} = 0$ for all v as claimed. \square

Proposition 4.3.3. *The image of the \mathbb{C}^\times -fixed point $\mathbf{X}_\lambda \in \text{CM}_n$ under ρ equals $\text{Res}_{\lambda^t}(q)$.*

Proof. The argument in the proof of Lemma 4.3.2 still works if we replace Z by $tI_n - Z$ where t is some indeterminant and we work over the field $\mathbb{C}(t)$. Therefore, Lemma 4.3.2 implies that $\det(tI_n - Z) = \prod_{a \in J} (t - a)$ where J is the multiset

$$\bigsqcup_{i=1}^l \{1, 2, \dots, r_i - 1, -(n_i - r_i), \dots, -2, -1, 0\},$$

when λ in Frobenius form is $(n_1, r_1), \dots, (n_l, r_l)$. Expressed in terms of the algebra $\mathbb{Z}[q^\kappa \mid \kappa \in \mathbb{C}]$, this is $\text{Res}_\lambda(q^{-1}) = \text{Res}_{\lambda^t}(q)$. \square

Remark 4.3.4. The reason for defining the map ρ is that Proposition 4.3.3 shows that ρ distinguishes the \mathbb{C}^\times -fixed points in CM_n . If $\lambda, \mu \vdash n$ and $\mathbf{X}_\lambda, \mathbf{X}_\mu$ the corresponding fixed points in CM_n then $\lambda = \mu$ if and only if $\rho(\mathbf{X}_\lambda) = \rho(\mathbf{X}_\mu)$.

4.4. Tangent spaces. In this section we calculate the graded character of the tangent space of CM_n at the \mathbb{C}^\times -fixed points. As noted in [34, Proposition 1.7], the differential of μ at $p = (X, Y; v, w)$ is given by

$$d_p \mu(P, Q, a, b) = [X, Q] + [P, Y] - v \circ b - a \circ w,$$

and this differential is surjective for all $p \in \widetilde{\text{CM}}_n$. Since the action of GL_n is free on $\widetilde{\text{CM}}_n$ we have a short exact sequence

$$0 \longrightarrow \mathfrak{g} \xrightarrow{d_e i} \mathfrak{g} \oplus \mathfrak{g} \oplus V \oplus V^* \xrightarrow{d_p \mu} \mathfrak{g} \longrightarrow 0, \quad (4.4.1)$$

where $d_e i$ is the differential of the action map $GL_n \rightarrow \widetilde{\text{CM}}_n$, $g \mapsto g \cdot p$, at the identity $e \in GL_n$. This differential $d_e i$ is given by $A \mapsto ([A, X], [A, Y], Av, -wA)$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of n . The Young diagram Y_λ of λ is the set

$$\{(i, j) \in \mathbb{Z}^2 \mid 0 \leq j \leq \ell(\lambda) - 1, 0 \leq i \leq \lambda_j - 1\},$$

visualized as in (4.2). Let q and t be indeterminants and define

$$B_\lambda(q, t) = \sum_{(a,b) \in Y_\lambda} q^b t^b.$$

Also for each $x \in Y_\lambda$ let $a(x)$ (resp $l(x)$), the *arm* of x (resp. the *leg* of x), be the number of boxes strictly to the right (resp. strictly above) x in Y_λ . For instance if $\lambda = (5, 4, 3, 3)$ as in (4.2) and $x = (0, 1)$ then $a(x) = 3$ and $l(x) = 2$. The *hook length* of x is defined to be $h(x) = a(x) + l(x) + 1$. The following combinatorial identity is shown as part of the proof of [31, Proposition 5.8].

Lemma 4.4.2. *Let λ be a partition of n , then*

$$(q + t - 1 - qt)B_\lambda(q, t)B_\lambda(q^{-1}, t^{-1}) + qtB_\lambda(q, t) + B_\lambda(q^{-1}, t^{-1}) = \sum_{x \in Y_\lambda} q^{1+a(x)}t^{-l(x)} + q^{-a(x)}t^{1+l(x)}.$$

Fix a partition λ of n and let $(X, Y; v, w)$ be the representative of the corresponding point in CM_n as described in 4.2. Set $q_i := n_1 + \dots + n_{i-1} + r_i$ and for each $\alpha \in \mathbb{C}^\times$ define the matrices $D = \text{diag}(1, \alpha, \dots, \alpha^{n-1})$ and $E = \bigoplus_i \alpha^{-q_i} I_{n_i}$. Then it is shown in the proof of [34, Proposition 6.11] that

$$\alpha^{-1} \text{Ad}_{ED}(X) = X, \quad \alpha \text{Ad}_{ED}(Z) = Z, \quad \alpha ED \cdot v = v, \quad w \cdot (ED)^{-1} \alpha^{-1} = w, \quad \forall \alpha \in \mathbb{C}^\times.$$

Therefore, if we define a twisted action of \mathbb{C}^\times on $\mathfrak{g} \times \mathfrak{g} \times V \times V^*$ by

$$\alpha \cdot (A, B; c, d) = (\alpha^{-1} \text{Ad}_{ED}(A), \alpha \text{Ad}_{ED}(B); \alpha ED \cdot c, d \cdot (ED)^{-1} \alpha^{-1}),$$

then $(X, Y; v, w) \in \widetilde{\text{CM}}_n$ is a \mathbb{C}^\times -fixed point. One should compare this with the idea of principal nilpotent pairs introduced in [17]. If we define an action of \mathbb{C}^\times on the other two copies of \mathfrak{g} by $\alpha \cdot A = \text{Ad}_{ED}(A)$ then the maps $d_e i$ and $d_p \mu$ are \mathbb{C}^\times -equivariant.

Proposition 4.4.3. *Let λ be a partition of n and $\mathbf{X}_\lambda = (X, Y; v, w)$ the corresponding fixed point in $\widetilde{\text{CM}}_n$. Then, the graded character of the tangent space of CM_n at \mathbf{X}_λ is given by*

$$\chi^{\mathbb{C}^\times}(T_{\mathbf{X}_\lambda} \text{CM}_n) = \sum_{x \in Y_\mu} q^{h(x)} + q^{-h(x)}.$$

Proof. Since the sequence (4.4.1) is exact and the maps \mathbb{C}^\times -equivariant,

$$\chi^{\mathbb{C}^\times}(T_{\mathbf{X}_\lambda} \text{CM}_n) = \chi^{\mathbb{C}^\times}(\mathfrak{g} \oplus \mathfrak{g} \oplus V \oplus V^*) - 2 \cdot \chi^{\mathbb{C}^\times}(\mathfrak{g}).$$

If W is a finite dimensional \mathbb{C}^\times -module then there is a natural action of \mathbb{C}^\times on $\text{End}_{\mathbb{C}}(W)$ and $\chi^{\mathbb{C}^\times}(\text{End}_{\mathbb{C}}(W)) = \chi^{\mathbb{C}^\times}(W; q) \chi^{\mathbb{C}^\times}(W; q^{-1})$. If we take V with the action of \mathbb{C}^\times given by $\eta \cdot v = ED \cdot v$, then the weights of V are just the exponents of entries of the diagonal matrix ED . This matrix is the diagonal sum of $n_i \times n_i$ matrices with diagonal exponents

$$-r_i, -r_i + 1, \dots, n_i - r_i - 1.$$

If we add one to each of these entries we get the content of the i^{th} strip of μ when it is written in Frobenius form. Therefore $\chi^{\mathbb{C}^\times}(V) = q^{-1} \text{Res}_\mu(q)$, which implies that

$$\chi^{\mathbb{C}^\times}(T_{\mathbf{X}_\lambda} \text{CM}_n) = (q + q^{-1} - 2) \cdot \text{Res}_\mu(q) \cdot \text{Res}_\mu(q^{-1}) + \text{Res}_\mu(q) + \text{Res}_\mu(q^{-1}). \quad (4.4.4)$$

The right hand side of equation (4.4.4) is the specialization $t \mapsto q^{-1}$ of

$$(q + t - 1 - qt)B_\mu(q, t)B_\mu(q^{-1}, t^{-1}) + qtB_\mu(q, t) + B_\mu(q^{-1}, t^{-1}).$$

Therefore, the proposition follows from Lemma 4.4.2 and the fact that the specialization $t \mapsto q^{-1}$ of

$$\sum_{x \in Y_\lambda} q^{1+a(x)}t^{-l(x)} + q^{-a(x)}t^{1+l(x)}$$

is just $\sum_{x \in Y_\mu} q^{h(x)} + q^{-h(x)}$. □

4.5. Factorization of the Calogero-Moser space. Let \mathfrak{h} be the subalgebra of diagonal matrices in \mathfrak{g} . By Chevalley's isomorphism, we identify $\mathfrak{g}/GL_n = \mathfrak{h}/\mathfrak{S}_n$. Let $\varpi : CM_n \rightarrow \mathfrak{h}/\mathfrak{S}_n$ be the map that sends the pair (X, Y) onto the GL_n -orbit of X . Similarly, let $\pi : CM_n \rightarrow \mathfrak{h}^*/\mathfrak{S}_n$ be the map that sends (X, Y) to the GL_n -orbit of Y . As the reader may notice, we have already used π and ϖ in the previous section. Since the isomorphism $X_c(\mathfrak{S}_n) \simeq CM_n$ of Proposition 5.2.1 intertwines these map, the choice of notation should not cause confusion. For $\mathbf{b} \in \mathfrak{h}^*/\mathfrak{S}_n$, the reduced fiber $\pi^{-1}(\mathbf{b})_{\text{red}}$ is denoted $CM(\mathbf{b})$. Write $\mathbf{b} = \sum_{i=1}^k n_i b_i$ with $b_i \in \mathbb{C}$ pairwise distinct and $\sum_i n_i = n$. If $(X, Y) \in CM(\mathbf{b})$, then we can decompose $Y = \bigoplus_{i=1}^k Y_i$, with Y_i an $n_i \times n_i$ matrix with only one eigenvalue b_i . We get a corresponding decomposition of $X = \bigoplus_{i=1, j}^k X_{i,j}$ and $v = \sum_i v_i$, $w = \sum_j w_j$. It is shown in the proof of [34, Lemma 6.3] that each $(X_{i,i}, Y_{i,i}; v_i, w_i)$ defines a point in $CM(n_i \cdot b_i)$. Thus, we have a map

$$\alpha_{\mathbf{b}} : CM(\mathbf{b}) \rightarrow \prod_{i=1}^k CM(n_i \cdot b_i).$$

Lemma 7.1 of [34] states:

Lemma 4.5.1. *The map $\alpha_{\mathbf{b}}$ is an isomorphism of affine varieties.*

Recall that the \mathbb{C}^\times -fixed points in CM_n are \mathbf{X}_λ , $\lambda \vdash n$. Define $\Omega_\lambda^{\text{cm}} := \{\mathbf{X} \in CM_n \mid \lim_{\alpha \rightarrow \infty} \mathbf{X} = \mathbf{X}_\lambda\}$. Then, it is shown in [34, Proposition 6.11] that

$$CM(n \cdot 0) = \bigsqcup_{\lambda \vdash n} \Omega_\lambda^{\text{cm}}. \quad (4.5.2)$$

If $\mathbf{b} = n \cdot b_1$ for some $b_1 \in \mathbb{C}$ then the map $(X, Y) \mapsto (X, Y + b_1 I_n)$ defines an isomorphism $CM(n \cdot 0) \simeq CM(\mathbf{b})$ and (4.5.2) implies that we get a decomposition of $CM(\mathbf{b})$ into cells $\Omega_{b_1, \lambda}^{\text{cm}}$. In general, if $\mathbf{b} = \sum_{i=1}^k n_i b_i$, then for every multipartition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(k)})$ of n such that $\lambda^{(i)} \vdash n_i$, define

$$\Omega_{\mathbf{b}, \lambda}^{\text{cm}} = \alpha_{\mathbf{b}}^{-1}(\Omega_{b_1, \lambda^{(1)}}^{\text{cm}} \times \dots \times \Omega_{b_k, \lambda^{(k)}}^{\text{cm}}).$$

Then, it follows from Lemma 4.5.1 and (4.5.2) that:

Proposition 4.5.3. *The space $CM(\mathbf{b})$ is a finite disjoint union of affine spaces*

$$CM(\mathbf{b}) = \bigsqcup_{\lambda} \Omega_{\mathbf{b}, \lambda}^{\text{cm}}$$

where the union is over all multipartitions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(k)})$ of n such that $\lambda^{(i)} \vdash n_i$.

5. RATIONAL CHEREDNIK ALGEBRAS OF TYPE A

In this section we consider rational Cherednik algebras at $t = 0$ associated to the symmetric group \mathfrak{S}_n . Since there is only one conjugacy class of reflections in \mathfrak{S}_n , the parameter \mathbf{c} is just a scalar $\mathbf{c} \in \mathbb{C}$. The rational Cherednik algebras for $\mathbf{c} \neq 0$ are all canonically isomorphic, therefore we may without loss of generality fix $\mathbf{c} = -2$. The centre of the corresponding rational Cherednik algebra is a regular algebra and we can apply the results of section 3.

5.1. The algebra $H_{\mathbf{c}}(\mathfrak{S}_n, \mathfrak{h})$. Let y_1, \dots, y_n be a basis of the n -dimensional space \mathfrak{h} and x_1, \dots, x_n a basis of \mathfrak{h}^* such that $x_i(y_j) = \delta_{i,j}$. The symmetric group \mathfrak{S}_n acts on \mathfrak{h} by permuting the y_i 's. The rational Cherednik algebra $H_n := H_{\mathbf{c}=-2}(\mathfrak{S}_n, \mathfrak{h})$ is the \mathbb{C} -algebra generated by \mathfrak{S}_n , \mathfrak{h} and \mathfrak{h}^* and satisfying the defining relations

$$\sigma x_i = x_{\sigma^{-1}(i)} \sigma, \quad \sigma y_i = y_{\sigma(i)} \sigma, \quad [x_i, x_j] = [y_i, y_j] = 0, \quad \forall 1 \leq i \neq j \leq n, \sigma \in \mathfrak{S}_n,$$

$$[y_i, x_j] = s_{ij}, \quad [y_i, x_i] = - \sum_{k=1, k \neq i}^n s_{ik}, \quad \forall 1 \leq i \neq j \leq n.$$

Note that $\mathbb{C}[x_1, \dots, x_n] = \mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[y_1, \dots, y_n] = \mathbb{C}[\mathfrak{h}^*]$. These relations come from relations (2.1.1) by taking $\alpha_{s_{ij}} = y_i - y_j$ and $\alpha_{s_{ij}}^\vee = x_i - x_j$, for all $1 \leq i \neq j \leq n$. The centre $Z(\mathrm{H}_c(\mathfrak{S}_n, \mathfrak{h}))$ of H_n will be written Z_n and the corresponding affine variety $X_n := \mathrm{Spec}(Z_n)$.

5.2. Relation to Wilson's Calogero-Moser space. Since Z_n is a regular algebra, there is a unique simple H_n -module supported at each closed point of X_n . For each such L , denote by χ_L the corresponding character of Z_n so that

$$z \cdot l = \chi_L(z) l, \quad \forall l \in L, z \in Z_n.$$

Thus, the map $L \mapsto \chi_L$ defines a bijection between $\mathrm{lrr}(\mathrm{H}_n)$ and the closed points of X_n . Each simple module L is isomorphic to the regular representation as an \mathfrak{S}_n -module. Therefore, if \mathfrak{S}_{n-1} is the subgroup of \mathfrak{S}_n acting on $\{2, \dots, n\}$ then the subspace $L^{\mathfrak{S}_{n-1}}$ is n -dimensional and x_1, y_1 act on this subspace.

Proposition 5.2.1 ([12], Theorem 11.16). *The map $L \mapsto (x_1|_{L^{\mathfrak{S}_{n-1}}}, y_1|_{L^{\mathfrak{S}_{n-1}}})$ defines an isomorphism of affine varieties $\psi_n : X_n \xrightarrow{\sim} \mathrm{CM}_n$.*

Remark 5.2.2. Recall that \mathbb{C}^\times acts on the spaces X_n and CM_n . The isomorphism ψ_n is \mathbb{C}^\times -equivariant. Recall from section 4.5 that we have maps $\pi : \mathrm{CM}_n \rightarrow \mathfrak{h}^*/\mathfrak{S}_n$ and $\varpi : \mathrm{CM}_n \rightarrow \mathfrak{h}/\mathfrak{S}_n$. It follows from the proof of [11, Theorem 10.21] that the following diagram commutes

$$\begin{array}{ccc} X_n & \xrightarrow{\psi_n} & \mathrm{CM}_n \\ & \searrow \varpi \times \pi & \swarrow \varpi \times \pi \\ & \mathfrak{h}/\mathfrak{S}_n \times \mathfrak{h}^*/\mathfrak{S}_n & \end{array}$$

5.3. Factorization. As was shown in (3.3), one can use completions of the rational Cherednik algebra to prove a factorization result for the generalized Calogero-Moser space X_n . On the other hand, as explained in (4.5), Wilson has shown that one can also factorize certain closed subvarieties of the classical Calogero-Moser space. In this section we show that these factorizations are compatible with the isomorphism of Proposition 5.2.1.

Fix $p \in \mathfrak{h}^*$ and denote its image in $\mathfrak{h}^*/\mathfrak{S}_n$ by $\mathbf{b} = \sum_{i=1}^k n_i \cdot b_i$. We may assume, without loss of generality, that $p = (b_1, \dots, b_1, b_2, \dots, b_2, b_3, \dots)$. The stabilizer of p with respect to \mathfrak{S}_n is $\mathfrak{S}_p := \mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_k}$. The rational Cherednik algebra $\mathrm{H}_p := \mathrm{H}_{c=-2}(\mathfrak{S}_p, \mathfrak{h})$ is isomorphic to a tensor product

$$\mathrm{H}_p \simeq \mathrm{H}_{n_1} \otimes \dots \otimes \mathrm{H}_{n_k}, \quad (5.3.1)$$

and hence

$$Z(\mathrm{H}_p) \simeq Z_{n_1} \otimes \dots \otimes Z_{n_k}. \quad (5.3.2)$$

Therefore, Corollary 3.3.7 implies that there is an isomorphism of affine varieties

$$\phi : \pi^{-1}(\mathbf{b})_{\mathrm{red}} \xrightarrow{\sim} \prod_{i=1}^k \pi^{-1}(n_i \cdot b_i)_{\mathrm{red}},$$

where $\pi^{-1}(n_i \cdot b_i)_{\mathrm{red}}$ is a closed subvariety in $\mathrm{Spec}(Z_{n_i})$. Recall the factorization of Wilson's Calogero-Moser space as described in Lemma 4.5.1.

Theorem 5.3.3. *The diagram*

$$\begin{array}{ccc} \pi^{-1}(\mathbf{b}) & \xrightarrow{\phi} & \prod_{i=1}^k \pi^{-1}(n_i \cdot b_i) \\ \psi_n \downarrow & & \downarrow \times_i \psi_{n_i} \\ \mathrm{CM}(\mathbf{b}) & \xrightarrow{\alpha_p} & \prod_{i=1}^k \mathrm{CM}(n_i \cdot b_i) \end{array}$$

is commutative.

Before we can give the proof of Theorem 5.3.3, we need to describe the isomorphism ϕ in representation theoretic terms. Firstly, since the diagram of the theorem involves isomorphisms between affine varieties it suffices to show commutativity on the level of closed points. Recall from (3.3) that we have the idempotent $e_1 \in \widehat{\mathbb{C}[\mathfrak{h}^*]}_{\mathfrak{p}}$, and if L is a simple H_n -module whose support is contained in $\pi^{-1}(\mathbf{b})$ then Proposition 3.3.6 says that $e_1 L$ is an irreducible H_p -module. Therefore, ϕ can be described as the map that takes the character χ_L of Z_n to the character $\chi_{e_1 L}$ of $Z_{n_1} \times \cdots \times Z_{n_k}$, for each simple H_n -module L whose support is contained in $\pi^{-1}(\mathbf{b})$.

Proof. To avoid any ambiguity, the generators of H_p will be denote $\hat{x}_1, \dots, \hat{x}_n, \hat{y}_1, \dots, \hat{y}_n$, as oppose to the generators of H_n , which are denoted x_1, \dots, x_n and y_1, \dots, y_n .

Let N be a simple H_p -module. Via the isomorphism (5.3.1), we write $N = N_1 \otimes \cdots \otimes N_k$, where N_i is a simple H_{n_i} -module. Define

$$m(i) = 1 + \sum_{r < i} n_r, \quad \forall 1 \leq i \leq k.$$

Then, the isomorphism $\mathrm{Spec}(Z(H_p)) \simeq \mathrm{CM}_{n_1} \times \cdots \times \mathrm{CM}_{n_k}$ that is induced from the factorization in (5.3.2) is given on the level of closed points by the map

$$N \mapsto [(X_1, Y_1), \dots, (X_k, Y_k)],$$

where X_i denotes the action of $\hat{x}_{m(i)}$ on $N_i^{\mathfrak{S}_{n_i-1}}$ and Y_i denotes the action of $\hat{y}_{m(i)}$ on $N_i^{\mathfrak{S}_{n_i-1}}$. Fix

$$W_i = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_{i-1}} \times \cdots \times \mathfrak{S}_{n_k}$$

so that, since N is the regular representation as an \mathfrak{S}_p -module, one can identify $N_i^{\mathfrak{S}_{n_i-1}} = N^{W_i}$.

Now let L be a simple H_n -module such that $\mathfrak{m}_{\mathbf{b}} \cdot L = 0$. Then, as explained above, the morphism ϕ can be described as taking χ_L to $\chi_{e_1 L}$. Therefore, to prove the commutativity of the diagram, we must show that if (X, Y) represent the action of x_1 and y_1 on $L^{\mathfrak{S}_{n-1}}$ with respect to some basis of that space then $(X_{i,i}, Y_{i,i})$ represent the action of $\hat{x}_{m(i)}, \hat{y}_{m(i)} \in H_p$ on $(e_1 L)^{W_i}$ with respect to some basis of that space.

Let $\mathbf{b} = \{p = p_1, \dots, p_l\}$ be the orbit of p under \mathfrak{S}_n . Since $\mathfrak{m}_{\mathbf{b}} \cdot L = 0$, we can decompose L with respect to the action of $\mathbb{C}[\mathfrak{h}]$ as

$$L = \bigoplus_{i=1}^l L_{p_i}.$$

Then, the functor e_1 sends L to $e_{1,1} L = L_{p_1}$ such that $x_i \cdot e_1 l = \hat{x}_i \cdot e_1 l$ for all $l \in L$. We can also decompose L with respect to the generalized eigenspaces of the action of y_1 :

$$L = \bigoplus_{i=1}^k L_{b_i}, \tag{5.3.4}$$

so that $Y_i : L_{b_i}^{\mathfrak{S}_{n-1}} \rightarrow L_{b_i}^{\mathfrak{S}_{n-1}}$ and $X_{i,j} : L_{b_j}^{\mathfrak{S}_{n-1}} \rightarrow L_{b_i}^{\mathfrak{S}_{n-1}}$. Let us fix an i . Let u_i denote the permutation in \mathfrak{S}_n that moves the block $[m(i), \dots, m(i+1) - 1]$ to $[1, \dots, n_i]$ and moves all the entries of $[1, \dots, n]$ below $m(i)$ up by n_i . Then, conjugation by u_i sends W_i into

$$\widetilde{W}_i = \mathfrak{S}_{n_i-1} \times \mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_k}.$$

We have $\widetilde{W}_i = \mathfrak{S}_{n-1} \cap \text{Stab}_{\mathfrak{S}_n}(u_i(p))$. Now

$$L_{b_i} = \bigoplus_{p_j \in I_i} L_{p_j}, \quad (5.3.5)$$

where $I_i = \{p_j | (p_j)_1 = b_i\}$. We have $u_i(p) \in I_i$ and \mathfrak{S}_{n-1} acts transitively on this set. This implies that $L_{u_i(p)} \subset L_{b_i}$ such that multiplication defines an isomorphism

$$\text{Ind}_{\widetilde{W}_i}^{\mathfrak{S}_{n-1}} L_{u_i(p)} \xrightarrow{\sim} L_{b_i}. \quad (5.3.6)$$

Hence, we have an explicit isomorphism

$$\rho : L_{u_i(p)}^{\widetilde{W}_i} \xrightarrow{\sim} L_{b_i}^{\mathfrak{S}_{n-1}}, \quad \rho(l) = \frac{1}{(n-1)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \sigma(l).$$

Recall that we want to compare the action of $X_{i,i}$ and $Y_{i,i}$ on $L_{b_i}^{\mathfrak{S}_{n-1}}$ with the action of $\hat{x}_{m(i)}, \hat{y}_{m(i)} \in H_p$ on $(e_1 L)^{W_i} = L_p^{W_i}$. Since $u_i(x_{m(i)}) = x_1$ and $u_i(y_{m(i)}) = y_1$, it suffices to consider the action of $\hat{x}_1, \hat{y}_1 \in H_p$ on $u_i(L_p^{W_i}) = L_{u_i(p)}^{\widetilde{W}_i}$. Thus, the theorem will follow from the following claim.

Claim 5.3.7. For all $l \in L_{u_i(p)}^{\widetilde{W}_i}$, we have

$$\rho(\hat{x}_1 l) = X_{i,i} \rho(l), \quad \rho(\hat{y}_1 l) = Y_{i,i} \rho(l).$$

Proof. The action of $X_{i,i}$ and $Y_{i,i}$ on $L_{b_i}^{\mathfrak{S}_{n-1}}$ is given by

$$X_{i,i} : L_{b_i}^{\mathfrak{S}_{n-1}} \xrightarrow{x_1 \cdot} L \xrightarrow{\text{pr}_i} L_{b_i}^{\mathfrak{S}_{n-1}}, \quad Y_{i,i} : L_{b_i}^{\mathfrak{S}_{n-1}} \xrightarrow{y_1 \cdot} L \xrightarrow{\text{pr}_i} L_{b_i}^{\mathfrak{S}_{n-1}},$$

where pr_i is projection onto $L_{b_i}^{\mathfrak{S}_{n-1}}$. Since multiplication by $u_i(e_1)$ is projection onto $L_{u_i(p)}$, (5.3.6) implies that pr_i can be expressed as multiplication by $\frac{1}{(n-1)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \sigma(u_i(e_1))$. Therefore,

$$\rho(\hat{x}_1 l) = \rho(x_1 \cdot u_i(e_1) l) = \frac{1}{(n-1)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \sigma(x_1 \cdot u_i(e_1) l) \quad (5.3.8)$$

$$= x_1 \frac{1}{(n-1)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \sigma(u_i(e_1) l) \quad (5.3.9)$$

A direct calculation, using the fact that the set $\{\sigma(u_i(e_1)) \mid \sigma \in \mathfrak{S}_{n-1}\}$ consists of orthogonal idempotents, shows that

$$\frac{1}{(n-1)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \sigma(u_i(e_1) l) = \left(\frac{1}{(n-1)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \sigma(u_i(e_1)) \right) \frac{1}{(n-1)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \sigma(u_i(e_1) l).$$

Therefore, we have

$$\begin{aligned}
(5.3.9) &= x_1 \cdot \left(\frac{1}{(n-1)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \sigma(u_i(e_1)) \right) \frac{1}{(n-1)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \sigma(u_i(e_1)l) \\
&= \left(\frac{1}{(n-1)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \sigma(u_i(e_1)) \right) x_1 \cdot \left(\frac{1}{(n-1)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \sigma(u_i(e_1)l) \right) \\
&= \text{pr}_i(x_1 \rho(l)) = X_{i,i} \cdot \rho(l)
\end{aligned}$$

as required. The proof of $\rho(\hat{y}_1 l) = Y_{i,i} \rho(l)$ is identical. \square

The statement of the theorem follows from the above claim. \square

5.4. Degenerate affine Hecke algebra. The fact that the degenerate affine Hecke algebra is a subalgebra of the rational Cherednik algebra of type **A** is well-known and has been extensively used to study the representation theory of rational Cherednik algebras at $t = 1$ e.g. [7] and [20]. Recently, Martino [25] has shown that this embedding of the degenerate affine Hecke algebra is also extremely useful at $t = 0$. Here, we will use it to gain better control of the isomorphism $X_n \xrightarrow{\sim} \text{CM}_n$. In particular, it will be used to write down the bijection between the \mathbb{C}^\times -fixed points in X_n and the fixed points in CM_n which is induced by the isomorphism $\psi_n : X_n \xrightarrow{\sim} \text{CM}_n$.

Definition 5.4.1. The *degenerate affine Hecke algebra* \mathcal{H}_n is the associative algebra generated by $\mathbb{C}[z_1, \dots, z_n]$ and \mathfrak{S}_n , satisfying the defining relations

$$s_i z_j = z_j s_i, \quad s_i z_i = z_{i+1} s_i - 1,$$

for all i and $j \neq i, i+1$, where $s_i := s_{i,i+1}$.

We note that the defining relations imply that $z_i s_i = s_i z_{i+1} - 1$. Also, as vector spaces, $\mathcal{H}_n \simeq \mathbb{C}[z_1, \dots, z_n] \otimes \mathbb{C}\mathfrak{S}_n$ and the centre of \mathcal{H}_n is the subalgebra $\mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}_n}$ of symmetric functions in the z_i 's, see [23]. The following lemma is a direct calculation.

Lemma 5.4.2. *The map*

$$z_i \mapsto y_i x_i + \sum_{j < i} s_{i,j} = x_i y_i - \sum_{j > i} s_{i,j}, \quad \forall 1 \leq i \leq n, \quad (5.4.3)$$

and $w \mapsto w$ for all $w \in \mathfrak{S}_n$ defines an embedding $\mathcal{H}_n \hookrightarrow H_n$ such that

$$[z_i, x_j] = \begin{cases} x_j s_{i,j} & j > i \\ x_i s_{i,j} & j < i \\ -\sum_{k < i} x_i s_{i,k} - \sum_{k > i} x_k s_{i,k} & i = j. \end{cases}$$

From now on we consider \mathcal{H}_n as a subalgebra of H_n . Remarkably, [25, Theorem 3.4] says that

Proposition 5.4.4. *The centre $\mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}_n}$ of \mathcal{H}_n is contained in Z_n .*

The embedding $\mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}_n} \hookrightarrow Z_n$ defines a morphism $\rho : X_n \rightarrow \mathbb{C}^n / \mathfrak{S}_n$ and, as in (4.3), we consider the points in $\mathbb{C}^n / \mathfrak{S}_n$ as defining elements of $\mathbb{Z}[q^\kappa \mid \kappa \in \mathbb{C}]$.

As in the definition of Verma modules for H_n , a standard tool in the study of the representation theory of \mathcal{H}_n is induction from representations of $\mathbb{C}[z_1, \dots, z_n]$. Therefore, for $a \in \mathbb{C}^n$, define

$$M(a) := \mathcal{H}_n \otimes_{\mathbb{C}[z_1, \dots, z_n]} a,$$

where a is considered a character of $\mathbb{C}[z_1, \dots, z_n]$ via evaluation. The module $M(a)$ is isomorphic to the regular representation as an \mathfrak{S}_n -module. Let \mathcal{D} be the dense, open subset of \mathbb{C}^n consisting

of all points $a = (a_1, \dots, a_n)$ such that $a_i - a_j \neq 0, \pm 1$ for all $1 \leq i \neq j \leq n$. Then, it is shown in [23, Lemma 6.1.2] that $M(a)$ is an irreducible \mathcal{H}_n -module for all $a \in \mathcal{D}$.

Lemma 5.4.5. *There exists a dense open subset U of X_n such that each irreducible \mathcal{H}_n -module L , whose support is contained in U , is isomorphic to $M(a)$ as a \mathcal{H}_n -module, for some $a \in \mathcal{D}$. In particular, each such L is irreducible as a \mathcal{H}_n -module.*

Proof. Let $U = \rho^{-1}(\overline{\mathcal{D}}) \subset X_n$, where $\overline{\mathcal{D}}$ denotes the image of \mathcal{D} in $\mathbb{C}^n/\mathfrak{S}_n$. Since $\overline{\mathcal{D}}$ is open in $\mathbb{C}^n/\mathfrak{S}_n$, U is open in X_n . The PBW theorems for \mathcal{H}_n and \mathcal{H}_n imply that the morphism ρ is dominant. Therefore, there exists a dense open subset U' of $\mathbb{C}^n/\mathfrak{S}_n$ such that $U' \subset \rho(X_n)$. Thus, $U' \cap \mathcal{D} \neq \emptyset$ implies that U is non-empty and hence dense in X_n because X_n is irreducible. Let L be a simple \mathcal{H}_n -module whose support is in U . Choose $v \in L$ to be a joint eigenvector for z_1, \dots, z_n . If a_1, \dots, a_n are the corresponding eigenvalues of the z_i 's then $a = (a_1, \dots, a_n) \in \mathcal{D}$ and $1 \otimes a \mapsto v$ defines a non-zero \mathcal{H}_n -module homomorphism $M(a) \rightarrow L$. This is an isomorphism because $\dim M(a) = \dim L$ and $M(a)$ is irreducible. \square

Lemma 5.4.6. *Let L be a simple \mathcal{H}_n -module such that $L \simeq M(a)$ with $a \in \mathbb{C}^n$ as a \mathcal{H}_n -module. Then the eigenvalues of $z_1 \in \mathcal{H}_n$ on $L^{\mathfrak{S}_{n-1}}$ are a_1, \dots, a_n .*

Proof. Since L is isomorphic to the regular representation as a \mathfrak{S}_n -module, a basis of $L^{\mathfrak{S}_{n-1}}$ is given by $\{e_0 s_{1,i} \otimes a \mid 1 \leq i \leq n\}$, where e_0 is the trivial idempotent in $\mathbb{C}\mathfrak{S}_{n-1}$. The lemma follows from a direct calculation which shows that action of z_1 on $L^{\mathfrak{S}_{n-1}}$ with respect to this basis is given by the matrix

$$z_1 = \begin{pmatrix} a_1 & -1 & \dots & -1 \\ 0 & a_2 & & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & a_n \end{pmatrix}.$$

Note that $s_{1,i} = s_1 \cdots s_{i-2} s_{i-1} s_{i-2} \cdots s_1 \in \mathfrak{S}_n$. Inductively, one can show that

$$z_1 s_1 \cdots s_{i-2} s_{i-1} = s_1 \cdots s_{i-2} s_{i-1} z_i - \sum_{j=1}^{i-1} s_1 \cdots \widehat{s}_j \cdots s_{i-1},$$

where $\widehat{\bullet}$ is used to denote omission. Similarly,

$$z_i s_{i-1} s_{i-2} \cdots s_1 = s_{i-1} s_{i-2} \cdots s_1 z_1 + \sum_{j=1}^{i-1} s_{i-1} \cdots \widehat{s}_j \cdots s_1.$$

Therefore, $z_1 s_{1,i} = s_{1,i} z_i - \sum_{j=1}^{i-1} s_1 \cdots \widehat{s}_j \cdots s_{i-1} s_{i-2} \cdots s_1$. Now, for $j < i-1$,

$$\sum_{j=1}^{i-1} s_1 \cdots \widehat{s}_j \cdots s_{i-1} s_{i-2} \cdots s_1 = (1, i, j+1),$$

where $(1, i, j+1)$ denotes a permutation written in cycle notation, and $s_1 \cdots \widehat{s}_{i-1} s_{i-2} \cdots s_1 = 1$. Hence

$$z_1 s_{1,i} = s_{1,i} z_i - 1 - \sum_{j=1}^{i-2} (1, i, j+1).$$

For each i, j , there exists some k such that $e_0(1, i, j+1) = e_0 s_{1,k}$. If we write

$$e_0 s_{1,i} = \frac{1}{(n-1)!} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(i)=1}} \sigma,$$

then clearly $k = j + 1$. Thus, $z_1 e_0 s_{1,i} = e_0 s_{1,i} z_i - e_0 - \sum_{j=2}^{i-1} e_0 s_{1,j} = e_0 s_{1,i} z_i - \sum_{j=1}^{i-1} e_0 s_{1,j}$. This gives the matrix form of z_1 described above. \square

Proposition 5.4.7. *The following diagram is commutative*

$$\begin{array}{ccc} X_n & \xrightarrow{\psi_n} & CM_n \\ & \searrow \rho & \swarrow \rho \\ & \mathbb{C}^n / \mathfrak{S}_n & \end{array}$$

Proof. Since ψ_n is an isomorphism, it suffices to show that there is a dense open subset U of X_n on which the diagram is commutative. We take U to be the subset of X_n described in Lemma 5.4.5. Each point in U is labeled by an irreducible H_n -module L such that $L \simeq M(a)$ with $a \in \mathcal{D}$ as a \mathcal{H}_n -module. The point χ_L labeled by L is sent by ψ_n to the pair (X, Y) , where $X = x_1|_{L^{\mathfrak{S}_{n-1}}}$ and $Y = y_1|_{L^{\mathfrak{S}_{n-1}}}$. Thus,

$$z_1|_{L^{\mathfrak{S}_{n-1}}} = y_1 x_1|_{L^{\mathfrak{S}_{n-1}}} = YX = Z.$$

By definition (4.3), $\rho \circ \psi_n(\chi_L)$ equals the eigenvalues of Z , which by Lemma 5.4.6 are a_1, \dots, a_n . On the other hand, $\rho(\chi_L)$ is the joint spectrum of z_1, \dots, z_n on $M(a)$, which is a_1, \dots, a_n , because $\mathbb{C}[z_1, \dots, z_n]^{\mathfrak{S}_n}$ is central in \mathcal{H}_n . \square

Theorem 5.4.8. *The isomorphism $\psi_n : X_n \xrightarrow{\sim} CM_n$ sends the \mathbb{C}^\times -fixed point $\chi_{L(\lambda)} \in X_n$ to the \mathbb{C}^\times -fixed point \mathbf{X}_λ in CM_n .*

Proof. As noted in remark 4.3.4, the map ρ distinguishes the fixed points. Therefore, by Proposition 5.4.7, it suffices to show that $\rho(\chi_{L(\lambda)}) = \rho(\mathbf{X}_\lambda) \in \mathbb{C}^n / \mathfrak{S}_n$. By Proposition 4.3.3, $\rho(\mathbf{X}_\lambda) = \text{Res}_{\lambda^t}(q)$. To calculate $\rho(\chi_{L(\lambda)})$, we need to calculate how the symmetric polynomials in the variables z_i act on $L(\lambda)$. Let $w_0 \in \mathfrak{S}_n$ be the longest word and $\Theta_i = \sum_{j < i} s_{i,j} \in \mathbb{C}\mathfrak{S}_n$ the i th Jucys-Murphy element. Then, as noted in [25, (5.4)], we have $-\sum_{j > i} s_{i,j} = -w_0 \Theta_i w_0$. Therefore, expression (5.4.3) for the z_i together with the arguments given in [25, (5.4)] imply that $\rho(\chi_{L(\lambda)}) = \text{Res}_\lambda(q^{-1})$, which equals $\text{Res}_{\lambda^t}(q)$. \square

5.5. Wilson's bispectral involution. There is a natural anti-involution $B : H_n \rightarrow H_n^{op}$ on the rational Cherednik algebra, extending the involution $\sigma \mapsto \sigma^{-1}$ on the group algebra $\mathbb{C}\mathfrak{S}_n$. It is defined by $B(x_i) = y_i$, $B(y_i) = x_i$ and $B(s_{ij}) = s_{ij}$. This allows us to define an auto-equivalence on $H_n\text{-mod}_{f.d.}$, the category of finite dimensional H_n -modules,

$$B : H_n\text{-mod}_{f.d.} \xrightarrow{\sim} H_n\text{-mod}_{f.d.}, \quad B(M) = M^*,$$

where M^* is the vector space dual and $(h \cdot f)(m) = f(B(h) \cdot m)$ for $h \in H_n$, $m \in M$ and $f \in M^*$. The anti-involution B restricts to an automorphism of Z_n and hence of X_n .

On the other hand, Wilson defined the *bispectral involution* b on G^{ad} , which in terms of Baker functions is given by $\tilde{\psi}_W(z, x) = \tilde{\psi}_{b(W)}(x, z)$. As noted in [34, page 4], the bispectral involution on CM_n is defined by $b(X, Y) = (Y^t, X^t)$. As one might expect, we have

Lemma 5.5.1. *We have $\psi_n \circ B = b \circ \psi_n$.*

Proof. Let L be a simple H_n -module and (X, Y) the matrices representing the action of (x_1, y_1) on $L^{\mathfrak{S}_{n-1}}$ with respect to some fixed basis. Then, with respect to the dual basis, the action of (y_1, x_1) on $(L^{\mathfrak{S}_{n-1}})^*$ is given by (Y^t, X^t) . \square

Recall (5.4) the \mathbb{C}^\times -fixed points $\mathbf{X}_\lambda \in CM_n$. The following observation is contained in [18].

Lemma 5.5.2. *For all $\lambda \vdash n$, $B(L(\lambda)) = L(\lambda)$. Thus, $B(\mathbf{X}_\lambda) = \mathbf{X}_\lambda$.*

5.6. Fourier transform. The Fourier transform, as introduced in [12, §4], is an automorphism of order four $F : H_n \xrightarrow{\sim} H_n$ define by

$$F(x_i) = y_i, \quad F(y_i) = -x_i, \quad F(w) = w, \quad \forall i \in [1, n], \quad w \in \mathfrak{S}_n.$$

We can use F to twist representations of H_n . If M is a H_n -module then, as a vector space, ${}^F M = M$ and the action of H_n on ${}^F M$ is defined by $h \cdot m = F(h)m$.

Lemma 5.6.1. *Choose $p \in \mathfrak{h}^*$, $q \in \mathfrak{h}$, $\mathbf{a} \in \mathfrak{h}/W$ and $\mathbf{b} \in \mathfrak{h}^*/W$.*

(1) *We have*

$$\begin{aligned} {}^F \Delta(p, \lambda) &= \nabla(p, \lambda), \quad {}^F \Delta(p, \lambda, \mathbf{a}) = \nabla(p, \lambda, -\mathbf{a}), \\ {}^F \nabla(q, \mu) &= \Delta(-q, \mu), \quad {}^F \nabla(q, \mu, \mathbf{b}) = \Delta(-q, \mu, \mathbf{b}). \end{aligned}$$

(2) *Let λ be a partition of n . Then, ${}^F L(\lambda) \simeq L(\lambda')$.*

Proof. Part (1) follows from the fact that $F(\mathbb{C}[\mathfrak{h}]) = F(\mathbb{C}[\mathfrak{h}^*])$ and F acts as the identity on $\mathbb{C}\mathfrak{S}_n$.

By part (1), ${}^F \Delta(0, \lambda, 0) \simeq \overline{H}_n \otimes_{\mathbb{C}[\mathfrak{h}]^{co\mathfrak{S}_n} \rtimes \mathfrak{S}_n} \lambda$, where \overline{H}_n is the restricted rational Cherednik algebra. As a $\mathbb{C}[\mathfrak{h}^*]^{co\mathfrak{S}_n} \rtimes \mathfrak{S}_n$ -module, ${}^F \Delta(0, \lambda, 0) \simeq \mathbb{C}[\mathfrak{h}^*]^{co\mathfrak{S}_n} \otimes \lambda$. The socle of this module is $\det(\mathbf{y}) \otimes \lambda$, where $\det(\mathbf{y}) = \prod_{i < j} (y_i - y_j)$. Since $\det(\mathbf{y}) \otimes \lambda \simeq \lambda'$ as an \mathfrak{S}_n -module and $\mathfrak{h} \cdot \det(\mathbf{y}) \otimes \lambda = 0$, it follows that there exists a non-zero homomorphism $\Delta(0, \lambda', 0) \rightarrow {}^F \Delta(0, \lambda, 0)$. The image of this homomorphism is contained in the socle of ${}^F \Delta(0, \lambda, 0)$, therefore it must factor through $L(\lambda')$, the simple head of $\Delta(0, \lambda', 0)$. The composition factors of ${}^F \Delta(0, \lambda, 0)$ are all isomorphic (since $\Delta(0, \lambda, 0)$ also has this property). Hence all these factors must be $L(\lambda')$. Applying F to the short exact sequence

$$0 \rightarrow \text{Ker} \rightarrow \Delta(0, \lambda, 0) \rightarrow L(\lambda) \rightarrow 0$$

shows that ${}^F L(\lambda) \simeq L(\lambda')$. \square

5.7. Adjoint anti-automorphism. Define the anti-automorphism $(-)^* : H_n \xrightarrow{\sim} H_n^{op}$ by $x_i^* = -x_i$, $y_i^* = y_i$ and $s_{i,j}^* = s_{i,j}$. As in (5.5), this defines an auto-equivalence

$$(-)^* : H_n\text{-mod}_{f.d.} \xrightarrow{\sim} H_n\text{-mod}_{f.d.}, \quad (M)^* = M^*,$$

where M^* is the vector space dual and $(h \cdot f)(m) = f(h^* \cdot m)$ for $h \in H_n$, $m \in M$ and $f \in M^*$. We also have the corresponding automorphism $(-)^*$ of X_n . Define the automorphism $(-)^*$ of CM_n by $(X, Y) \mapsto (-X^t, Y^t)$ and recall from section 6.1 that $W \mapsto W^*$ defines an automorphism $(-)^*$ of G^{ad} .

Lemma 5.7.1. *We have $(-)^* \circ \psi_n = \psi_n \circ (-)^*$ and $(-)^* \circ \beta_n = \beta_n \circ (-)^*$.*

Proof. The proof of the first statement is completely analogous to the proof of Lemma 5.5.1. The second statement is [34, Lemma 7.7]. \square

Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(k)})$ be a multipartition. The transpose of λ is defined componentwise, $\lambda^t = ((\lambda^{(1)})^t, \dots, (\lambda^{(k)})^t)$.

Proposition 5.7.2. *Under the adjoint automorphism,*

$$\Omega_{\mathbf{b}, \lambda}^* = \Omega_{\mathbf{b}, \lambda'}.$$

Proof. By Lemma 5.7.1, we can work either with the rational Cherednik algebra or in the Calogero-Moser space. First, we note that one can deduce from the explicit formula for $(-)^*$ on CM_n , together with the factorization construction given by Wilson, section 4.5, that we have

$$(\Omega_{\mathbf{b}, \lambda}^{\text{cm}})^* = \alpha_{\mathbf{b}}^{-1}((\Omega_{b_1, \lambda^{(1)}}^{\text{cm}})^* \times \dots \times (\Omega_{b_k, \lambda^{(k)}}^{\text{cm}})^*).$$

Moreover, for $b \in \mathbb{C}$ and $t_b : \text{CM}_n \xrightarrow{\sim} \text{CM}_n$, $t_b(X, Y) = (X, Y - bI_n)$ we have

$$(\Omega_\lambda^{\text{cm}})^* = [t_b(\Omega_{b,\lambda}^{\text{cm}})]^* = t_b([\Omega_{b,\lambda}^{\text{cm}}]^*).$$

Therefore, it suffices to show that $(\Omega_\lambda^{\text{cm}})^* = \Omega_{\lambda'}^{\text{cm}}$. The automorphism $(-)^*$ is also \mathbb{C}^\times -equivariant. Hence, it suffices to show that $\mathbf{X}_\lambda^* = \mathbf{X}_{\lambda'}$. For this, we use the fact that $(-)^* = F \circ B$. Therefore, the result follows from Lemmata 5.5.2 and 5.6.1. \square

6. GRASSMANNIANS

In [34], Wilson constructed an embedding of the Calogero-Moser space into the adelic Grassmannian, a certain infinite dimensional (non-algebraic!) space. This embedding will allow us to identify the support of the endomorphism algebra of Verma modules for the rational Cherednik algebra with Schubert cells. Defining this embedding requires the use of several auxiliary infinite dimensional Grassmannians, which we will also require. In order to facilitate the reader in keeping track of all these Grassmannians, we list them here with reference to where they are first defined in the text. We have

$$G^{\text{Ad}} \xrightarrow{\sim} G^{\text{ad}} \hookrightarrow \mathcal{G}^{\text{rat}} \subset \tilde{\mathcal{G}}^{\text{rat}},$$

where

- G^{Ad} is the Adelic Grassmannian (6.1.2),
- G^{ad} is the adelic Grassmannian (6.1.4),
- \mathcal{G}^{rat} is the reduced rational Grassmannian (6.1.1),
- and
- $\tilde{\mathcal{G}}^{\text{rat}}$ is the rational Grassmannian (6.1.1).

We also have another pair of infinite dimensional Grassmannians, the canonical Grassmannian $\mathcal{Q}\mathcal{E}$, defined in (6.4.3), which is contained inside the quasi-exponential Grassmannian $\mathcal{Q}\text{Gr}$, defined in (6.3.3). The Grassmannians G^{Ad} , G^{ad} and $\mathcal{Q}\mathcal{E}$ can all be realized as an infinite union of finite dimensional spaces

$$G^{\text{Ad}} = \bigsqcup_{n=1}^{\infty} G_n^{\text{Ad}}, \quad G^{\text{ad}} = \bigsqcup_{n=1}^{\infty} G_n^{\text{ad}}, \quad \mathcal{Q}\mathcal{E} = \bigsqcup_{n=1}^{\infty} \mathcal{Q}\mathcal{E}_n,$$

and we have identifications $G^{\text{Ad}} \xrightarrow{\sim} G^{\text{ad}} \xleftarrow{\sim} \mathcal{Q}\mathcal{E}$ which restrict to

$$\text{CM}_n \xrightarrow{\sim} G_n^{\text{Ad}} \xrightarrow{\sim} G_n^{\text{ad}} \xleftarrow{\sim} \mathcal{Q}\mathcal{E}_n.$$

Finally, in section 6.8, we will also consider the relative Grassmanian $\mathcal{G}_n^{\text{rel}}$ and comment on the embedding $\text{CM}_n \xrightarrow{\sim} \mathcal{Q}\mathcal{E}_n \hookrightarrow \mathcal{G}_n^{\text{rel}}$.

It is possible to equip most of the above spaces with topologies, making the maps between them continuous. Since this fact will not play a role in what we do, it will be easier for us simply to think of them as sets.

At various stages, we will define “Schubert cells” in each Grassmannian. The notation used to denote these cells depends on which Grassmannian they sit inside, namely

$$\Omega_\bullet \subset X_n, \quad \Omega_\bullet^{\text{cm}} \subset \text{CM}_n, \quad \Omega_\bullet^{\text{ad}} \subset G^{\text{ad}}, \quad \Omega_\bullet^{\text{qe}} \subset \mathcal{Q}\text{Gr}.$$

6.1. The adelic Grassmannian. In this section we recall the definition of the Adelic Grassmannian G^{Ad} and the adelic Grassmannian G^{ad} . Before we can do this we need to define the rational Grassmannian.

Definition 6.1.1. The *rational Grassmannian* $\tilde{\mathcal{G}}^{\text{rat}}$ is the space of all \mathbb{C} -subspaces W of the field $\mathbb{C}(z)$ such that

- (1) there exist polynomials $a(z), b(z) \in \mathbb{C}[z]$ such that $a(z)^{-1}\mathbb{C}[z] \supseteq W \supseteq b(z)\mathbb{C}[z]$;

(2) we have $\dim a(z)^{-1}\mathbb{C}[z]/W = \deg(a)$.

The *reduced rational Grassmannian* \mathcal{G}^{rat} is defined to be the proper subset of $\tilde{\mathcal{G}}^{\text{rat}}$ consisting of those spaces W such that one can chose $a(z) = b(z)$ in the above definition.

The Adelic Grassmannian is defined in a similar manner: For each $b \in \mathbb{C}$, let Gr_b be the Grassmannian of all subspaces W of $\mathbb{C}(z)$ such that

- (1) there exist some $k \geq 0$ with $(z-b)^{-k}\mathbb{C}[z] \supseteq W \supseteq (z-b)^k\mathbb{C}[z]$;
- (2) we have $\dim(z-b)^{-k}\mathbb{C}[z]/W = k$.

The space $\mathbb{C}[z]$ belongs to Gr_b for all $b \in \mathbb{C}$.

Definition 6.1.2. The *Adelic Grassmannian* is defined to be the restricted product

$$G^{\text{Ad}} := \prod_{b \in \mathbb{C}}^0 \text{Gr}_b,$$

where $\{W_b\}_{b \in \mathbb{C}}$ belongs to G^{Ad} if and only if $W_b = \mathbb{C}[z]$ for all but finitely many $b \in \mathbb{C}$.

The support of $\{W_b\} \in G^{\text{Ad}}$ is the finite subset of \mathbb{C} consisting of all b such that $W_b \neq \mathbb{C}[z]$. It is clear that each Gr_b is a subspace of \mathcal{G}^{rat} . This can be extended to an embedding of the whole of G^{Ad} into \mathcal{G}^{rat} . For $b \in \mathbb{C} \cup \{\infty\}$, define the symmetric bilinear form $\langle f, g \rangle_b = \text{res}_{z=b} f(z)g(z)dz$ on $\mathbb{C}(z)$. The annihilator of a subspace W of $\mathbb{C}(z)$ with respect to this form is written

$$\text{Ann}_b W = \{f \in \mathbb{C}(z) \mid \langle f, g \rangle_b = 0 \forall g \in W\}.$$

The annihilator $\text{Ann}_\infty W$ will be denoted W^* . As noted in [34, §2.2], the involution $W \mapsto W^*$ preserves each of the subsets Gr_b (this not true of the other $\text{Ann}_b -$). Define the embedding $i : G^{\text{Ad}} \rightarrow \mathcal{G}^{\text{rat}}$ by

$$i(\{W_b\}) = \bigcap_{b \in \mathbb{C}} \text{Ann}_b(W_b^*). \quad (6.1.3)$$

Definition 6.1.4. The image of the i inside \mathcal{G}^{rat} is called the *adelic Grassmannian* and denoted G^{ad} .

That i is indeed an embedding is shown in [34, Lemma 2.5]. One can check directly that the restriction of i to Gr_b is just the naive inclusion $\text{Gr}_b \subset \mathcal{G}^{\text{rat}}$. Let $W \in \text{Gr}_b$. Then, by definition, there exists some $N \gg 0$ such that $(z-b)^N\mathbb{C}[z] \subset W \subset (z-b)^{-N}\mathbb{C}[z]$ and $\dim W/(z-b)^N\mathbb{C}[z] = N$. Thus, $W/(z-b)^N\mathbb{C}[z]$ belongs to $\text{Gr}_N((z-b)^{-N}\mathbb{C}[z]/(z-b)^N\mathbb{C}[z])$. There is a natural stratification of $\text{Gr}_N((z-b)^{-N}\mathbb{C}[z]/(z-b)^N\mathbb{C}[z])$ into Schubert cells (to be recalled below) labelled by all partitions that fit into an $N \times N$ box. Then, $W/(z-b)^N\mathbb{C}[z]$ will belong to a particular cell, labeled by λ say. We define the degree of W to be $|\lambda|$. One can easily check that this definition is independent of the choice of N . Moreover, since the degree of $\mathbb{C}[z] \in \text{Gr}_b$ is 0, the definition extends additively to the whole of G^{Ad} . Let G_n^{Ad} be the set of all spaces of degree n and G_n^{ad} the image of G_n^{Ad} under i . There is another characterization of the space G_n^{ad} in terms of the τ -function, see section 6.6. Namely, G_n^{ad} is the set of all W in G^{ad} such that $\tau_W(t_1, 0, 0, \dots)$ is a polynomial of degree n .

The action of \mathbb{C}^\times on $\mathbb{C}(z)$ given by $\alpha \cdot z = \alpha^{-1}z$ induces an action of \mathbb{C}^\times on G^{Ad} and \mathcal{G}^{rat} , making i equivariant.

6.2. One of the key results of [34] is the construction of an embedding of the Calogero-Moser space into the adelic Grassmannian. Since this construction is rather technical, we will not recall the details, but simply note the features that we will require.

Theorem 6.2.1 ([34], Theorem 1). *There is an embedding $\beta_n : \text{CM}_n \rightarrow G^{\text{ad}}$, whose image is G_n^{ad} .*

We define $\text{Supp} : G_n^{\text{Ad}} \rightarrow \mathfrak{h}^*/\mathfrak{S}_n$ by

$$\text{Supp}(\{W_b\}) = \sum_{b \in \mathbb{C}} \deg(W_b) \cdot b,$$

which via i may also be considered as a map $G_n^{\text{ad}} \rightarrow \mathfrak{h}^*/\mathfrak{S}_n$. By [34, Theorem 7.5], the following diagram commutes

$$\begin{array}{ccc} \text{CM}_n & \xrightarrow{\beta_n} & G_n^{\text{ad}} \\ & \searrow \pi & \swarrow \text{Supp} \\ & \mathfrak{h}^*/\mathfrak{S}_n & \end{array} \quad (6.2.2)$$

Next we wish to describe a partition of G^{ad} into Schubert cells. We begin by considering the spaces $W \in \text{Gr}_0 \subset G^{\text{ad}}$ i.e. those spaces $W \in G^{\text{ad}}$ such that $\text{Supp}(W) = \deg(W) \cdot 0$. If $W \in \text{Gr}_0$ then there exists an integer k such that $z^{-k}\mathbb{C}[z] \supset W \supset z^k\mathbb{C}[z]$. Therefore, we can chose a basis

$$\left\{ z^{s_i} + \sum_{j=s_i+1}^{s_{i+1}-1} \alpha_{i,j} z^j \mid i \in \mathbb{N} \right\} \quad (6.2.3)$$

of W such that $s_i = i$ for $i \gg 0$. As in [32, §3], a basis $\{w_i\}_{i \in \mathbb{N}_0}$ of W is said to be *admissible* if $w_i = z^i$ for $i \gg 0$. The set (6.2.3) is an admissible basis. If we associate to each w_i , the degree s_i of the trailing term of w_i , then we get a set $S_W = \{s_0, s_1, \dots\}$. The set S satisfies $s_i = i$ for $i \gg 0$ and each such set corresponds to a partition λ , defined by $\lambda_i = i - s_i$ so that $\lambda_0 \geq \lambda_1 \geq \dots$ and $\lambda_i = 0$ for $i \gg 0$. The \mathbb{C}^\times -fixed points in Gr_0 of the action defined in section 6.1 are

$$W_\lambda = \text{Span} \{z^s \mid s \in S\}$$

where S is the set corresponding to λ . Then,

$$\text{Gr}_0 = \bigsqcup_{\lambda \in \mathcal{P}} \Omega_\lambda^{\text{ad}}$$

where $\Omega_\lambda^{\text{ad}} = \{W \in \text{Gr}_0 \mid \lim_{\alpha \rightarrow \infty} \alpha \cdot W = W_\lambda\}$ is a Schubert cell in Gr_0 . It is the set of all spaces W such that $S_W = \lambda$.

For $b \in \mathbb{C}$, let $t_b : \mathbb{C}(z) \rightarrow \mathbb{C}(z)$ be the automorphism $z \mapsto z - b$. Then, t_b defines an isomorphism $\text{Gr}_0 \xrightarrow{\sim} \text{Gr}_b$ and we set $\Omega_{b,\lambda}^{\text{ad}} = t_b(\Omega_\lambda^{\text{ad}})$. Now let $\mathbf{b} = \sum_{i=1}^k n_i b_i \in \mathfrak{h}^*/\mathfrak{S}_n$ and $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ a multipartition of n such that $\lambda^{(i)} \vdash n_i$. We define

$$\Omega_{\mathbf{b},\boldsymbol{\lambda}}^{\text{ad}} = \left\{ i(\{W_{b_i}\}) \mid \{W_{b_i}\} \in \prod_{i=1}^k \Omega_{b_i,\lambda^{(i)}}^{\text{ad}} \right\}.$$

Lemma 6.2.4. *For each $\mathbf{b} \in \mathbb{C}^n/\mathfrak{S}_n$ and $\boldsymbol{\lambda}$ multipartition of type \mathbf{b} , we have $\beta_n(\Omega_{\mathbf{b},\boldsymbol{\lambda}}^{\text{cm}}) = \Omega_{\mathbf{b},\boldsymbol{\lambda}}^{\text{ad}}$.*

Proof. The diagram (6.2.2) implies that it suffices to show that $\beta_n(\Omega_\lambda^{\text{cm}}) = \Omega_\lambda^{\text{ad}}$. Since β_n is \mathbb{C}^\times -equivariant and both $\Omega_\lambda^{\text{cm}}$ and $\Omega_\lambda^{\text{ad}}$ are defined to be attracting sets for the \mathbb{C}^\times -action, it suffices to show that $\beta_n(\mathbf{X}_\lambda) = W_\lambda$. This is shown in [34, Proposition 6.13]. \square

6.3. Quasi-exponentials. Recall from the introduction that \mathcal{Q} denotes the space of all functions of the form $\sum_{i=1}^k e^{b_i x} g_i(x)$, where $b_i \in \mathbb{C}$ and $g_i(x) \in \mathbb{C}[x]$. We think of the space \mathcal{Q} as being a space of linear functionals on the vector space $\mathbb{C}[z]$ via the pairing

$$\langle e^{bx} g(x), f(z) \rangle = e^{b\partial} g(\partial) \cdot f(z)|_{z=0}, \quad (6.3.1)$$

where, formally, $e^{b\partial} \cdot z^n = (z+b)^n$. The pairing $\langle -, - \rangle$ satisfies $\langle x \cdot c, f \rangle = \langle c, \partial_x f \rangle$ and $\langle \partial_x \cdot c, f \rangle = \langle c, z f \rangle$. There is also a \mathbb{C}^\times -action on \mathcal{Q} given by $\alpha \cdot x = \alpha x$. The pairing $\langle -, - \rangle$ is \mathbb{C}^\times -invariant.

Definition 6.3.2. A finite dimensional subspace C of \mathcal{Q} is said to be a *space of quasi-exponentials*. A quasi-exponential $f \in \mathcal{Q}$ is said to be *homogeneous* if $f = e^{bx} g(x)$ for some $b \in \mathbb{C}$ and $g(x) \in \mathbb{C}[x]$. A space of quasi-exponentials C is said to be *homogeneous* if

$$C = \bigoplus_{b \in \mathbb{C}} C_b,$$

where C_b consists entirely of homogeneous quasi-exponentials of the form $e^{bx} g(x)$ for some $b \in \mathbb{C}$ and $g(x) \in \mathbb{C}[x]$.

Definition 6.3.3. The set of all homogeneous spaces of quasi-exponentials is called the *quasi-exponential Grassmannian* and denoted \mathcal{QGr} .

We have $\mathcal{QGr} = \bigsqcup_{n=0}^{\infty} \mathcal{QGr}_n$, where \mathcal{QGr}_n is the set of all homogeneous spaces of quasi-exponentials of dimension n . We define $\text{Supp} : \mathcal{QGr}_n \rightarrow \mathbb{C}^n / \mathfrak{S}_n$ by $\text{Supp}(C) = \sum_{i=1}^k n_i \cdot b_i$ if $C = \bigoplus_{i=1}^k C_{b_i}$ with $\dim C_{b_i} = n_i$. As shown in [33, Proposition 4.6], the spaces of quasi-exponentials are related to the rational Grassmannian as follows. For $C \subset \mathcal{Q}$, define

$$V_C := \{f \in \mathbb{C}[z] \mid \langle g, f \rangle = 0, \forall g \in C\}.$$

Lemma 6.3.4. *The subspace $W \subset \mathbb{C}(z)$ belongs to $\tilde{\mathcal{G}}^{\text{rat}}$ if and only if there exists a finite dimensional subspace $C \subset \mathcal{Q}$ and polynomial q with $\deg(q) = \dim C$ such that $W = q^{-1}V_C$.*

Proof. Fix $C \subset \mathcal{Q}$ with $\dim C < \infty$ and $q \in \mathbb{C}[z]$ such that $\deg q = \dim C$. Then there exist $b_1, \dots, b_n \in \mathbb{C}$ and $r_1, \dots, r_n \in \mathbb{N}$ such that $C \subset \text{Span} \{e^{b_i x} x^{r_i}\}$. The polynomial

$$h = \prod_{i=1}^n (z - b_i)^{r_i+1}$$

has the property that $\langle e^{b_i x} x^{r_i}, h f \rangle = 0$ for all $1 \leq i \leq n$ and all $f \in \mathbb{C}[z]$. Therefore, the ideal $h \mathbb{C}[z]$ is contained in V_C and hence $h q \mathbb{C}[z] \subset V_C$ as well. Thus,

$$h \mathbb{C}[z] = (h q) q^{-1} \mathbb{C}[z] \subset q^{-1} V_C \subset q^{-1} \mathbb{C}[z],$$

which implies that $q^{-1} V_C \in \tilde{\mathcal{G}}^{\text{rat}}$. To prove the converse, we note that if $u = \prod_{i=1}^n (z - b_i)^{a_i}$ then

$$C_u = \{g \in \mathcal{Q} \mid \langle g, u f \rangle = 0 \forall f \in \mathbb{C}[z]\} = \text{Span} \{e^{(a_i - 1) \lambda_i}\}.$$

In particular, $\dim C_u = \deg u$. The pairing $\langle -, - \rangle$ identifies $C_u = (\mathbb{C}[z]/(u))^*$. Given $p \mathbb{C}[z] \subset W \subset q^{-1} \mathbb{C}[z]$, we have $p q \mathbb{C}[z] \subset q W \subset \mathbb{C}[z]$. Set $u = p q$ and let $C = \{\mu \mid \langle \mu, f \rangle = 0 \forall f \in q W\}$. Then, the identification $C_u \simeq (\mathbb{C}[z]/(u))^*$ induces an isomorphism $\text{Gr}_k(C_u) \xrightarrow{\sim} \text{Gr}_{\deg u - k}(\mathbb{C}[z]/(u))$, under which $q W = V_C$. Since $\text{codim}_{\mathbb{C}[z]}(q W) = \deg q$, we have

$$\dim C = \dim(\overline{q W})^\perp = \text{codim}_{\mathbb{C}[z]/(u)}(\overline{q W}) = \text{codim}_{\mathbb{C}[z]}(q W) = \deg q.$$

□

If $C \in \mathcal{QGr}$ is a homogeneous space of quasi-exponentials then define

$$q_C(z) = \prod_{b \in \text{Supp}(C)} (z - b)^{n_b}$$

where $n_b = \dim C_b$. Then $\deg(q_C) = \dim C$ and, by Lemma 6.3.4, $q_C^{-1} V_C$ is a point in $\tilde{\mathcal{G}}^{\text{rat}}$. Write $\gamma : \mathcal{QGr} \rightarrow \tilde{\mathcal{G}}^{\text{rat}}$ for the map $C \mapsto q_C^{-1} V_C$. As explained in [33, §6],

Proposition 6.3.5. *The image of the map γ equals \mathcal{G}^{ad} .*

Unfortunately, as noted in [33, §6], the set of all homogeneous spaces of quasi-exponentials does not map bijectively onto G^{ad} .

6.4. Canonical spaces. For each $W \in G^{\text{ad}}$, there is a canonical choice of a space C in the fiber $\gamma^{-1}(W)$. This choice allows us to define a subset of \mathcal{QGr} such that the restriction of γ to this subset is a bijection.

Definition 6.4.1. Let $C \subset \mathcal{Q}$ be a homogeneous space of quasi-exponentials and fix a homogeneous basis $e^{b_1 x} g_1(x), \dots, e^{b_n x} g_n(x)$ of C . The *Wronskian* of C is defined to be

$$\text{Wr}_C(x) := \det (\partial_x^i (e^{b_j x} g_j(x)))_{i,j=1,\dots,n} \cdot e^{-\sum_{i=1}^n b_i x}. \quad (6.4.2)$$

The Wronskian is (up to a scalar) independent of the choice of basis and is a polynomial in x . The *degree* of C is defined to be $\deg(C) := \deg(\text{Wr}_C)$ and the space C is said to be *canonical* if $\dim C = \deg(C)$.

Definition 6.4.3. The *canonical Grassmannian* is defined to be the set of all canonical, homogeneous spaces of quasi-exponentials. It is denoted \mathcal{QE} .

We first show that there is a unique canonical space in $\gamma^{-1}(W)$ for all $W \in \text{Gr}_0 \subset G^{\text{ad}}$. Recall from section 6.2 that we have a partition of Gr_0 into Schubert cells $\Omega_\lambda^{\text{qe}}$. Let $W \in \Omega_\lambda^{\text{qe}}$ and define $S = \{s_0, s_1, \dots\}$ as in (6.2). We can multiply W by z^N for some $N \geq -s_0 = \lambda_0$ so that $z^N W \subset \mathbb{C}[z]$ and then take the annihilator C of this space in \mathcal{Q} .

Lemma 6.4.4. Let $W \in \text{Gr}_0$ be of degree n and let r be the smallest positive integer such that $z^r W \subset \mathbb{C}[z]$. For each $N \geq r$ set $C_N = \text{Ann}_{\mathcal{Q}} z^N W$. Then, C_n is the unique canonical space of quasi-exponentials in the set $\{C_N \mid N \geq r\}$.

Proof. Let λ be a partition of n and assume that $W \in \Omega_\lambda^{\text{qe}}$. Then, $r = \lambda_0$. For any $N \geq \lambda_0$, the space C_N is homogeneous because $z^d \mathbb{C}[z] \subset z^N W$ for some d implies that C_N consists entirely of polynomials in x . We claim that $\deg \text{Wr}_{C_N}(x) = n$ for all $N \geq \lambda_0$. By definition, this claim is equivalent to the statement of the lemma. Therefore, we will give a proof of the claim. Let $r_i = s_i + N$ so that

$$z^{r_i} + \sum_{j=d_i+1}^{r_{i+1}-1} \alpha_{i,j} z^j$$

is a basis for $z^N W$. We claim that the number of elements in $\mathbb{N}_0 \setminus (S + N)$ is N . To see this, consider the set $S + N$ as a collection of beads on \mathbb{N} . Moving all beads as far right as possible gives us the set $N + \mathbb{N}$. In doing so this the number of gaps does not change. Then, the claim follows from the obvious fact that $|\mathbb{N} \setminus (N + \mathbb{N})| = N$. Write $\{e_0 < e_1 < \dots < e_{N-1}\}$ for $\mathbb{N} \setminus (S + N)$. Then, C_N has a basis given by

$$x^{e_i} + \sum_{\substack{j=0 \\ j \neq e_k, \forall k}}^{e_i-1} \beta_{i,j} x^j.$$

Here, x is the linear functional such that $\langle x^k, f \rangle = \frac{1}{k!} \partial^k(f)|_{z=0}$ so that $\langle x^k, x^l \rangle = \delta_{k,l}$. Recall that we have chosen $d \gg 0$ such that $z^d \mathbb{C}[z] \subset z^N W$. The degree of the Wronskian of C_N is $\sum_{i=0}^{N-1} [e_i - (N - 1 - i)]$. We need to calculate this number. First, note that $\{0, 1, \dots, N + d - 1\} = \{r_0, \dots, r_{d-1}\} \sqcup \{e_0, \dots, e_{N-1}\}$ so that

$$\sum_{i=0}^{N+d-1} i = \sum_{j=0}^{d-1} r_i + \sum_{i=0}^{N-1} e_i$$

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Hence

$$\frac{(N+d)(N+d-1)}{2} - \sum_{j=0}^{d-1} r_j = \sum_{i=0}^{N-1} e_i,$$

equivalently,

$$\frac{(N+d)(N+d-1)}{2} - \left(\sum_{j=0}^{d-1} j - s_j \right) - \left(\sum_{j=0}^{d-1} N + j \right) = \sum_{i=0}^{N-1} e_i.$$

Thus, $\frac{N(N-1)}{2} + n = \sum_{i=0}^{N-1} e_i$ which implies that $\sum_{i=0}^{N-1} e_i - (N-i) = n$ as claimed. In fact, one can see that making N larger just makes the tail $\{x^0, x^1, x^2, \dots\}$ of the basis of C_N longer and doesn't affect the Wronskian. \square

Using the fact that $\partial_x^k e^{bx} g(x) = e^{bx} (\partial_x + b)^k g(x)$, one can check that the same argument applies to any space $W \in \text{Gr}_b$. That is, if $W \in \Omega_{b,\lambda}^{\text{qe}}$ where $\lambda \vdash n$, then $\text{Ann}_{\mathcal{Q}}(z-b)^n W$ is the unique canonical space in the fiber $\gamma^{-1}(W)$. Therefore, we define $\eta : \text{G}^{\text{Ad}} \rightarrow \mathcal{QE}$ by

$$\eta(\{W_b\}) = \bigoplus_{b \in \mathbb{C}} \text{Ann}_{\mathcal{Q}} \left[(z-b)^{\deg(W_b)} W_b \right].$$

The map η is a bijection.

Proposition 6.4.5. *The diagram*

$$\begin{array}{ccc} \text{G}^{\text{Ad}} & \xrightarrow{\eta} & \mathcal{QE} \\ & \searrow i \quad \swarrow \gamma & \\ & \text{G}^{\text{ad}} & \end{array}$$

is commutative.

Proof. Let $W = \{W_b\} \in \text{G}^{\text{ad}}$ with $\eta(\{W_b\}) = \bigoplus_{b \in \text{Supp}(W)} C_b$. The commutativity of the diagram is the statement

$$\left(\prod_{b \in \text{Supp}(W)} (z-b)^{-n_b} \right) \text{Ann}_{\mathbb{C}[z]} \left(\bigoplus_{b \in \text{Supp}(W)} C_b \right) = \bigcap_{b \in \mathbb{C}} \text{Ann}_b(W_b^*).$$

Since $\text{Ann}_{\mathbb{C}[z]}(C_b) = (z-b)^{n_b} W_b$, we must show that

$$\bigcap_{b \in \text{Supp}(W)} (z-b)^{n_b} W_b = \left(\prod_{b \in \text{Supp}(W)} (z-b)^{n_b} \right) \bigcap_{b \in \mathbb{C}} \text{Ann}_b(W_b^*).$$

Since $(z-b)^{n_b} W_b \subset \mathbb{C}[z]$ for all $b \in \text{Supp}(W)$, we may rewrite the above as

$$\bigcap_{b \in \mathbb{C}} (z-b)^{n_b} W_b = \left(\prod_{b \in \mathbb{C}} (z-b)^{n_b} \right) \bigcap_{b \in \mathbb{C}} \text{Ann}_b(W_b^*), \quad (6.4.6)$$

where $n_b = 0$ for b not in the support of W . Let LHS refer to the left hand side of equation (6.4.6) and RHS to the right hand side of (6.4.6). We first show that the LHS is contained in the RHS. For all $b \in \mathbb{C}$, we have $W_b \subseteq \text{Ann}_b(W_b^*)$. In fact, by [34, Lemma 2.5], W_b is the subspace of $\text{Ann}_b(W_b^*)$ consisting of all functions whose only pole is at b . Let f belong to the LHS. Then $f \in (z-b)^{n_b} W_b$ and hence $(z-b)^{-n_b} f \in \text{Ann}_b(W_b^*)$ for all b . If we take any function $g \in \text{Ann}_b(W_b^*)$ and $h \in \mathbb{C}(z)$

such that h has no pole at b , then $gh \in \text{Ann}_b(W_b^*)$. This implies that $(\prod_{a \in \mathbb{C}} (z-a)^{-n_a})f \in \text{Ann}_b(W_b^*)$. Hence,

$$\left(\prod_{a \in \mathbb{C}} (z-a)^{-n_a} \right) f \in \bigcap_{b \in \mathbb{C}} \text{Ann}_b(W_b^*) \implies f \in \left(\prod_{b \in \mathbb{C}} (z-b)^{n_b} \right) \bigcap_{b \in \mathbb{C}} \text{Ann}_b(W_b^*).$$

Thus, LHS is contained in RHS.

Now assume that $f \in \bigcap_{b \in \mathbb{C}} \text{Ann}_b(W_b^*)$. Then, $\left(\prod_{\substack{a \in \mathbb{C} \\ a \neq b}} (z-a)^{n_a} \right) f$ belongs to $\text{Ann}_b(W_b^*)$ and has no poles other than at b . Therefore, [34, Lemma 2.5] implies that $\left(\prod_{\substack{a \in \mathbb{C} \\ a \neq b}} (z-a)^{n_a} \right) f$ belongs to W_b . Hence,

$$\left(\prod_{a \in \mathbb{C}} (z-a)^{n_a} \right) f \in (z-b)^{n_b} W_b, \forall b \in \mathbb{C} \implies \left(\prod_{a \in \mathbb{C}} (z-a)^{n_a} \right) f \in \bigcap_{b \in \mathbb{C}} (z-b)^{n_b} W_b.$$

Thus, RHS is contained in LHS. \square

Proposition 6.4.5 implies that there is a well-defined bijection $\eta \circ i^{-1} : G^{\text{ad}} \rightarrow \mathcal{QE}$. We will also denote this map by η .

6.5. Schubert Cells. We recall the standard definition of Schubert cells in $\text{Gr}_n(\mathbb{C}[x]_{2n}) \subset \mathcal{QE}$, where $\mathbb{C}[x]_{2n}$ denote the space of all polynomials in $\mathbb{C}[x]$ of degree less than $2n$, as given in [15, page 147]. Let

$$\mathcal{F} = \{0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{2n} = \mathbb{C}[x]_{2n}\}$$

be a complete flag in $\mathbb{C}[x]_{2n}$. Then, given a partition $\lambda = (\lambda_0, \dots, \lambda_{n-1})$ with at most n parts such that $\lambda_0 \leq n$, the Schubert cell $\Omega_\lambda(\mathcal{F}) \subset \text{Gr}_n(\mathbb{C}[x]_{2n})$ is given by

$$\Omega_\lambda(\mathcal{F}) = \{V \in \text{Gr}_n(\mathbb{C}[x]_{2n}) \mid \dim(V \cap F_k) = i \text{ for } n+i-\lambda_{i-1} \leq k \leq n+i-\lambda_i \text{ and all } 0 \leq i \leq n\},$$

where the condition for $i = 0$ is $\dim(V \cap F_{n-\lambda_0}) = 0$. Then, $\dim \Omega_\lambda(\mathcal{F}) = n^2 - |\lambda|$. The flag at infinity is

$$\mathcal{F}(\infty) = \{0 \subset \mathbb{C}[x]_1 \subset \mathbb{C}[x]_2 \subset \cdots \subset \mathbb{C}[x]_{2n}\}.$$

A partition λ with at most n parts such that $\lambda_0 \leq n$ is precisely the same as a partition that fits into an $n \times n$ square. The complement of λ in this square is the rotation by π of another partition, denoted $\bar{\lambda}$. It is the unique partition such that $\lambda_i + \bar{\lambda}_{n-i-1} = n$ for all $i = 0, 1, \dots, n-1$. For each partition λ of n , we define $\Omega_\lambda^{\text{qe}} := \Omega_{\bar{\lambda}}(\mathcal{F}(\infty))$. It is n -dimensional. The \mathbb{C}^\times -fixed point in $\Omega_\lambda^{\text{qe}}$ has basis $\{x^{d_i} \mid i = 0, \dots, n-1\}$, where $d_i = n + \lambda_i - (i+1)$.

Lemma 6.5.1. *Let λ be a partition of n , then $\eta(\Omega_\lambda^{\text{ad}}) = \Omega_{\lambda_t}^{\text{qe}}$.*

Proof. The proof of Lemma 6.4.4 shows that the map $\eta : W \mapsto \text{Ann}_{\mathcal{Q}}(z^n W)$ sends the Schubert cell $\Omega_\lambda^{\text{ad}}$ to the set U_λ consisting of all spaces in $\text{Gr}_n(\mathbb{C}[x]_{2n}) \subset \mathcal{QE}$ with basis

$$\left\{ x^{e_i} + \sum_{\substack{j=0 \\ j \neq e_k}}^{e_i-1} \beta_{i,j} x^j \mid 0 \leq i \leq n-1 \right\}. \quad (6.5.2)$$

Note that $\dim U_\lambda = \sum_{i=0}^{n-1} (e_i - i) = n$. If $V \in \text{Gr}_n(\mathbb{C}[x]_{2n})$ has a basis as in (6.5.2) then $\dim(V \cap \mathbb{C}[x]_k) = \#\{i \mid e_i < k\}$, which equals j say if and only if $e_{j-1} < k \leq e_j$. Therefore $U_\lambda = \Omega_{\bar{\mu}}(\mathcal{F}(\infty))$ where $\bar{\mu}$ is the partition given by $e_j = n + j - \bar{\mu}_j$. Equivalently, $e_{n-j-1} = 2n - (j+1) - \bar{\mu}_{n-j-1}$. Since

$\bar{\mu}$ is defined by $\mu_j + \bar{\mu}_{n-j-1} = n$, we see that $e_{n-j-1} = n + \mu_j - (j+1)$. Thus, from the definition of $\{r_i\}$ and $\{e_i\}$ given in the proof of Lemma 6.4.4, it follows that μ is the (unique) partition of n such that

$$\{0, 1, \dots, 2n-1\} = \{n+i-\lambda_i \mid 0 \leq i \leq n-1\} \sqcup \{n+\mu_j-(j+1) \mid 0 \leq j \leq n-1\}.$$

One can deduce that this implies that $\mu = \lambda^t$ from the fact that $\mathbb{Z} = S_\lambda \sqcup -S_{\lambda^t}$, which is easily checked. \square

We fix coordinates on the Schubert cell $\Omega_\lambda^{\text{qe}}$ by fixing basis

$$f_i(x) = x^{e_i} + a_{i,1}z^{e_i-1} + \dots + a_{i,0}, \quad \forall i = 0, 1, \dots, n-1$$

where $e_i = n + \lambda_i - (i+1)$ and $a_{i,j} \equiv 0$ if $e_i - j \in \{e_{i+1}, \dots, e_{n-1}\}$, for each $C \in \Omega_\lambda^{\text{qe}}$. Then, $\mathbb{C}[\Omega_\lambda^{\text{qe}}]$ is a polynomial ring in the $a_{i,j}$. Let $\mathbf{b} = \sum_{i=1}^k n_i b_i \in \mathfrak{h}^*/\mathfrak{S}_n$ and $\lambda = (\lambda^{(1)}, \dots, \lambda^{(k)})$ a multipartition of n such that $\lambda^{(i)} \vdash n_i$. Inside \mathcal{QGr} we have the product of Grassmannians

$$\text{Gr}_{\mathbf{b}}(\mathcal{QGr}) = \text{Gr}_{n_1} \left(e^{b_1 x} \mathbb{C}[x]_{2n_1} \right) \times \dots \times \text{Gr}_{n_k} \left(e^{b_k x} \mathbb{C}[x]_{2n_1} \right).$$

As usual, we define $\Omega_{\mathbf{b}, \lambda}^{\text{qe}}$ to be the product $\Omega_{b_1, \lambda^{(1)}}^{\text{qe}} \times \dots \times \Omega_{b_k, \lambda^{(k)}}^{\text{qe}}$ in $\text{Gr}_{\mathbf{b}}(\mathcal{QGr})$. The set $\text{Gr}_{\mathbf{b}}(\mathcal{QGr})$ has a natural scheme structure, such that $\Omega_{\mathbf{b}, \lambda}^{\text{qe}}$ is a locally closed subvariety. Moreover,

$$\text{Gr}_{\mathbf{b}}(\mathcal{QGr}) \cap \mathcal{QE} = \bigsqcup_{\lambda \vdash \mathbf{b}} \Omega_{\mathbf{b}, \lambda}^{\text{qe}}.$$

6.6. The τ function. The rational Grassmannian is a subspace of Sato's Grassmannian and therefore plays an important role in the study of the Kadomtsev-Petviashvili (KP) hierarchy. It also means that, via the Boson-Fermion correspondence, we can associate to each $W \in \mathcal{G}^{\text{rat}}$ its τ -function, which is a rational function in the infinitely many variables³ t_1, t_2, t_3, \dots

$$\tau_W(t_1, t_2, t_3, \dots) \in \mathbb{C}(t_1, t_2, t_3, \dots).$$

See [26] for the definition of τ_W . A more geometric definition of the τ -function in terms of a non-vanishing section of the dual of the determinant line bundle on \mathcal{G}^{rat} is given in [32]. One can also define τ -functions on the Calogero-Moser space CM_n and on the set of all spaces of quasi-exponentials in \mathcal{Q} as follows. Let $(X, Y) \in \text{CM}_n$ and define

$$\tau_{(X,Y)}(t_1, t_2, t_3, \dots) = \det \left(X + \sum_{i=1}^{\infty} i t_i (-Y)^{i-1} \right). \quad (6.6.1)$$

As shown in section 3.8 of [34], we have $\tau_{(X,Y)} = \tau_{\beta_n(X,Y)}$.

Let C be a space of quasi-exponential and fix a basis $\{c_1, \dots, c_n\}$ of this space. Define

$$\tau_C^0(t_1, t_2, t_3, \dots) = \det \left(\langle c_i, z^j G(z) \rangle \right)_{i,j=1 \dots n},$$

where $\langle -, - \rangle$ is the pairing (6.3.1) and $G(z) := \exp \left(\sum_{i=1}^{\infty} z^i t_i \right)$. Assume that $\text{Supp } C = \sum_{j=1}^k n_j b_j$ and define

$$\tau_C(t_1, t_2, \dots) = \left(\prod_{j=1}^k \exp \left(- \sum_{i=1}^{\infty} b_j^i t_i \right)^{n_j} \right) \tau_C^0(t_1, t_2, \dots).$$

Lemma 6.6.2. *For all $C = \eta(W)$ in \mathcal{QE} , we have $\tau_W = \tau_C$ and*

$$\text{Wr}_C(x) = \tau_C(x, 0, \dots). \quad (6.6.3)$$

³Often, in the literature, one sets $x = t_1$.

Proof. As shown in [33, (5.7)], if $\text{Supp} C = n \cdot 0$ then $\tau_W = \tau_C^0$, which obviously is the same as τ_C . The general formula will follow from [32, Lemma 3.8], for which we need to use the language of symmetric functions. Let Λ be the ring of symmetric functions and denote by p_i , resp. h_i, e_i , the i th power, resp. complete symmetric and elementary symmetric, function in Λ . If we proclaim (see [32, Proposition 8.2]) that

$$G(z)^{-1} = 1 + \sum_{i=1}^{\infty} h_i z^i := H(z),$$

then this forces $-it_i = p_i$, which is a consequence of the identity

$$\exp \left(\sum_{i=1}^{\infty} \frac{1}{i} \sum_{j=1}^{\infty} t_j^i z^i \right) = \prod_{j \geq 1} \exp \left(\sum_{i=1}^{\infty} \frac{1}{i} (t_j z)^i \right) = \prod_{j \geq 1} \frac{1}{(1 - t_j z)} = H(z).$$

Then, $G(z)$ equals $\prod_{j \geq 1} \exp \left(- \sum_{i=1}^{\infty} \frac{1}{i} (t_j z)^i \right) = E(-z)$, where

$$E(z) = 1 + \sum_{i=1}^{\infty} e_i z^i = \prod_{i \geq 1} (1 + t_i z)$$

is the generating function for the elementary symmetric functions. Set $g := G(z)$ and let

$$\tilde{g} = \prod_{j=1}^k \frac{1}{(1 - b_j z^{-1})^{n_j}}.$$

If we define f and \tilde{f} by $g = \exp(f)$ and $\tilde{g} = \exp(\tilde{f})$, then

$$\tilde{f} = \sum_{i=1}^{\infty} \frac{p_i(\mathbf{b})}{i} z^{-i}, \quad f = - \sum_{i=1}^{\infty} \frac{p_i}{i} z^i,$$

where $p_i(\mathbf{b}) = p_i(b_1, \dots, b_1, b_2, \dots, b_2, b_3, \dots, b_k, 0, \dots)$ with b_j occurring n_j times. Then,

$$S(\tilde{f}, f) := \frac{1}{2\pi i} \int_{S^1} \tilde{f}'(z) f(z) dz = \sum_{i=1}^{\infty} \frac{p_i(\mathbf{b})}{i} p_i.$$

Lemma 3.8 of [32] says that, after making the substitution $-it_i = p_i$, we have $\tau_C = \exp(S(\tilde{f}, f)) \tau_C^0$. Since

$$\exp \left(- \sum_{i=1}^{\infty} p_i(\mathbf{b}) t_i \right) = \prod_{j=1}^k \exp \left(- \sum_{i=1}^{\infty} b_j^i t_i \right)^{n_j},$$

the claim $\tau_W = \tau_C$ follows.

Recall the definition of $\text{Wr}_C(x)$ as given in (6.4.2). If one makes the substitution $t_1 = x$, $t_2 = t_3 = \dots = 0$ into τ_C then the equality (6.6.3) is evident. \square

It would be interesting to have a representation-theoretic interpretation of the τ -function for the rational Cherednik algebra.

6.7. Let $W \in \text{Gr}_0$ and fix some admissible basis $\{w_i\}_{i \in \mathbb{N}_0}$ of W . The admissible basis may be thought of as a $\mathbb{Z} \times \mathbb{N}$ matrix, where the columns are the vectors w_i . Then, $W \in \text{Gr}_n(z^{-n}\mathbb{C}[z]/z^n\mathbb{C}[z])$ if $w_i = z^{i-1}$ for all $i > n$. The corresponding matrix has the form

$$\begin{pmatrix} \cdots & \vdots & \cdots & \cdots & \vdots & \cdots \\ & 0 & & & 0 & \\ & \vdots & & & \vdots & \\ \cdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ w_{1,-n} & \cdots & w_{n,-n} & & \vdots & \\ \vdots & \ddots & \vdots & \cdots & 0 & \cdots \\ \cdots & \vdots & \cdots & \vdots & \vdots & \\ w_{1,n-1} & \cdots & w_{n,n-1} & & \vdots & \\ \cdots & \vdots & & 1 & 0 & \\ \cdots & 0 & \cdots & 0 & \ddots & \ddots \\ & \vdots & & & \ddots & 1 \end{pmatrix}$$

For each $\lambda = S \in \mathcal{P}$, the determinant $w^\lambda := w^S = \det(w_{i,j})_{i \in S, j \in \mathbb{N}}$ is well-defined. Also, if any $s_k \geq n$ for $k < n$ then $w^S = 0$, since the k th column of $(w_{i,j})_{i \in S, j \in \mathbb{N}}$ is the zero vector. Therefore, we may assume that $\{s_0, \dots, s_{n-1}\}$ is a subset of the interval $[-n, n-1]$. Thus, there are $\binom{2n}{n}$ such S . Since these determinants depend, up to a scalar, on a choice of admissible basis, this means that we have defined a map $\text{Gr}_n(z^{-n}\mathbb{C}[z]/z^n\mathbb{C}[z]) \rightarrow \mathbb{P}^{\binom{2n}{n}-1}$. This is nothing but the classical Plücker embedding. In terms of partitions, each coordinate of $\mathbb{P}^{\binom{2n}{n}-1}$ is labelled by a partition of length at most n such that $\lambda_0 \leq n$ i.e. all partitions that fit into a square of length n . The Plücker embedding is \mathbb{C}^\times -equivariant and the fixed points x_λ of the \mathbb{C}^\times -action on $\text{Gr}_n(z^{-n}\mathbb{C}[z]/z^n\mathbb{C}[z])$ are mapped to the points $w^\lambda = 1$ and $w^\nu = 0$ for all $\nu \neq \lambda$.

If, as in the proof of Lemma 6.6.2, we make the substitution $-it_i = p_i$, where p_i is the i th power polynomial in the ring Λ of symmetric functions, then the τ -function belongs to Λ . By [32, Proposition 8.2], the expansion of τ in terms of Schur polynomials

$$\tau_W = \sum_{\lambda \in \mathcal{P}} w^\lambda s_\lambda$$

has coefficients given by the determinants w^λ . Therefore, if $W \in \text{Gr}_n(z^{-n}\mathbb{C}[z]/z^n\mathbb{C}[z])$, then $\tau_W = \sum_{\lambda \in \square} w^\lambda s_\lambda$.

The map $\eta : \text{G}^{\text{ad}} \rightarrow \mathcal{QE}$ restricts to an isomorphism $\text{Gr}_n(z^{-n}\mathbb{C}[z]/z^n\mathbb{C}[z]) \xrightarrow{\sim} \text{Gr}_n(\mathbb{C}[x]_{2n})$, which sends V to $(z^n V)^\perp$. Thus, it is clearly an isomorphism of varieties. If $C \in \text{Gr}_n(\mathbb{C}[x]_{2n})$, then $\tau_C = \sum_{\lambda \in \square} c^\lambda s_\lambda$, where each c^λ is a homogeneous function on $\text{Gr}_n(\mathbb{C}[x]_{2n})$ which once again just defines the usual Plücker embedding.

Theorem 6.7.1. *The map $\eta \circ \beta_n : \text{CM}_n \rightarrow \mathcal{QE} \subset \mathcal{QGr}$ restricts to an isomorphism of algebraic varieties $\Omega_{b,\lambda}^{\text{cm}} \simeq \Omega_{b,\lambda}^{\text{qe}} \subset \text{Gr}_b(\mathcal{QGr})$.*

Proof. Since both η and β_n behave well with respect to factorization, by diagram (6.2.2) and Proposition 6.4.5, it suffices to show that $\eta \circ \beta_n : \text{CM}_n \rightarrow \mathcal{QE}$ restricts to an isomorphism of algebraic varieties $\Omega_{\lambda}^{\text{cm}} \simeq \Omega_{\lambda}^{\text{qe}} \subset \text{Gr}_{n,0}(\mathbb{C}[x]_{2n})$. We expand

$$\tau_{(X,Y)} = \sum_{\mu \in \mathcal{P}} f_\mu(X, Y) s_\mu,$$

where each $f_\lambda(X, Y) \in \mathbb{C}[\text{CM}_n]$. Define $\mathbb{C}[\Omega_{\lambda^t}^{\text{qe}}] \xrightarrow{\sim} \mathbb{C}[\Omega_{\lambda^t}^{\text{cm}}]$ by $c^\mu(a_{i,j}) = f_\mu(X, Y)$. That this is well-defined and that it is an isomorphism both follow from the fact that the pair of spaces $\Omega_{\lambda^t}^{\text{qe}}$ and $\Omega_{\lambda^t}^{\text{cm}}$ are reduced and that the τ -function distinguishes closed points of both spaces. \square

6.8. As noted in [13], one can interperate Wilson's embedding β_n as an embedding of CM_n into

$$\mathcal{G}_n^{\text{rel}} := \{(I, W) \mid I \triangleleft \mathbb{C}[z] \text{ with } \dim \mathbb{C}[z]/I = n \text{ and } W \subset \mathbb{C}[z]/I^2 \text{ an } n\text{-dimensional subspace.}\},$$

the *relative Grassmannian*. Since both CM_n and $\mathcal{G}_n^{\text{rel}}$ are quasi-projective varieties, it is natural to expect that Wilson's embedding is a morphism of varieties. In this subsection we suggest one way that one might hope to show this. Projection onto I defines a proper map $\mathcal{G}_n^{\text{rel}} \rightarrow \mathbb{A}^{(n)} = \text{Hilb}^n(\mathbb{C})$. Let E be the rank $2n$ vector bundle on $\mathbb{A}^{(n)}$, whose fiber over I is $\mathbb{C}[z]/I^2$. Recall, [22, Example 2.2.3], that the relative Grassmanian is the space that represents the contravariant functor $F : \text{Sch}_{\mathbb{A}^{(n)}} \rightarrow \text{Sets}$, from the category of schemes over $\mathbb{A}^{(n)}$ to sets defined by

$$F(X) = \{\phi : \xi^* E \rightarrow \mathcal{F} \mid \mathcal{F} \text{ flat of rank } n\} / \simeq.$$

where $\xi : X \rightarrow \mathbb{A}^{(n)}$.

We denote by R the coordinate ring of CM_n . Recall that $\pi : \text{CM}_n \rightarrow \mathfrak{h}^*/\mathfrak{S}_n$. Let $\mathcal{E} = \pi^* E$ be the vector bundle of rank $2n$ on CM_n induced by E . Since CM_n is affine, we consider instead the corresponding projective R -module of section, which we will also denote by \mathcal{E} . Since \mathcal{E} is the pull-back of a projective $\mathbb{C}[\mathbb{A}^{(n)}]$ -module, it is actually a free R -module. Explicitly,

$$\mathcal{E} = R[z]/(\det(z - Y)^2).$$

Associated to each space $W \in \text{G}^{\text{ad}}$ is the Baker function $\tilde{\psi}_W(z, x)$, see [33] and [34]. Just as for the τ -function, the Baker function distinguishes points in that $\tilde{\psi}_{W_1}(z, x) = \tilde{\psi}_{W_2}(z, x)$ if and only if $W_1 = W_2$. If $\text{Supp}(W) = \sum_{i=1}^k n_i b_i$, then define $\Psi_W(z) = \prod_{i=1}^k (z - b_i)^{n_i}$. The *regular* Baker function $\psi_W(z, x)$ is defined to be $\Psi_W(z) \tilde{\psi}_W(z, x)$. We define the *polynomial* Baker function to be

$$\psi_W^{\text{pol}}(z, x) = \text{Wr}_W(x) \cdot \psi_W(z, x) = \Psi_W(z) \cdot \text{Wr}_W(x) \cdot \tilde{\psi}_W(z, x).$$

It is known, e.g. as a consequence of [33, Proposition 6.5], that $\psi_W^{\text{pol}}(z, x) = g(z, x)e^{zx}$, where $g(z, x)$ is a polynomial of degree $\deg(W)$ in both z and x . The following lemma follows from the description of $\tilde{\psi}_W$ given in section 4 of [33].

Lemma 6.8.1. *Let $W \in \text{G}^{\text{ad}}$ and $C = \eta(W) \in \mathcal{QE}$. Then,*

$$\Psi_W(z)W = \text{Span} \{(\partial_x^k \psi_W^{\text{pol}}(z, x))|_{x=0} \text{ for all } k \in \mathbb{N}\} = C^\perp.$$

For each $(X, Y) \in \text{CM}_n$, consider the element $\psi_{(X,Y)}^{\text{pol}} = e^{xz} \det((X-x)(Y-z)-1) \in R \hat{\otimes} \mathbb{C}[[x, z]]$. Let \mathcal{K} be the R -submodule of \mathcal{E} generated by

$$\partial_x^0 \psi|_{x=0}^{\text{pol}}, \partial_x^1 \psi|_{x=0}^{\text{pol}}, \partial_x^2 \psi|_{x=0}^{\text{pol}}, \dots$$

Then, we define \mathcal{F} to be the quotient \mathcal{E}/\mathcal{K} .

Conjecture 6.8.2. The quotient $\mathcal{E} \rightarrow \mathcal{F}$ is a vector bundle of rank n on CM_n , inducing a locally closed embedding $\beta_n : \text{CM}_n \rightarrow \mathcal{G}_n^{\text{rel}}$.

Remark 6.8.3. The definition of CM_n as a G.I.T. quotient implies that there is a “tautological” rank n bundle on the space. It is unclear to the author how this tautological bundle is related to \mathcal{F} .

Expanding, $\psi_W^{\text{pol}}(z, x)e^{-zx} = \sum_{i,j=0}^n a_{i,j} z^i x^j$, we write

$$D_W = \sum_{i,j=0}^n a_{i,j} x^j \partial_x^i.$$

Lemma 6.8.4. *Let $W \in G^{\text{ad}}$. Then, $C = \eta(W) \in \mathcal{QE}$ is the space of all holomorphic solutions of the differential equation D_W .*

Proof. By Lemma 6.8.1, $\Psi_W(z)W = C^\perp$, which equals $\text{Span} \{(\partial_x^k \psi_W^{\text{pol}}(z, x))|_{x=0} \mid k = 0, 1, \dots\}$. We apply the easy identity $(\partial_x^k x^j e^{zx})|_{x=0} = \partial_z^j(z^k)$. Thus, $c \in C$ if and only if

$$\begin{aligned} \langle c, (\partial_x^k \psi_W(z, x))|_{x=0} \rangle &= \left\langle c, \sum_{i,j=0}^n a_{i,j} z^i (\partial_x^k (x^j) e^{zx})|_{x=0} \right\rangle \\ &= \left\langle c, \sum_{i,j=0}^n a_{i,j} z^i \partial_z^j(z^k) \right\rangle = \left\langle \sum_{i,j=0}^n a_{i,j} x^j \partial_x^i c, z^k \right\rangle = 0 \end{aligned}$$

for all $k \in \mathbb{N}$. This implies that $\sum_{i,j=0}^n a_{i,j} x^j \partial_x^i c = 0$. Since the dimension of C is n , C contains all solutions of the differential equation D_W . \square

7. SCHUBERT CALCULUS

7.1. Schubert Cells. In this subsection, we give a proof of Theorem 1.2.1. We define $\nu_n : X_n \rightarrow \mathcal{QGr}$ to be the composition $\eta \circ \beta_n \circ \psi_n$, so that ν_n identifies X_n with its image \mathcal{QE}_n in \mathcal{QGr} . Recall that Theorem 1.2.1 claims that ν_n restricts to an isomorphism of algebraic varieties

$$\nu_n : \Omega_{b,\lambda} \xrightarrow{\sim} \Omega_{b,\lambda}^{\text{qe}} \subset \text{Gr}_b(\mathcal{QGr}).$$

This statement will follow from Theorem 6.7.1, if we can show that $\psi_n(\Omega_{b,\lambda}) = \Omega_{b,\lambda}^{\text{cm}}$. By Theorem 5.3.3, ψ_n is compatible with factorizations. Therefore, it suffices to show that $\psi_n(\Omega_\lambda) = \Omega_\lambda^{\text{cm}}$ for λ a partition of n . Both Ω_λ and $\Omega_\lambda^{\text{cm}}$ are attracting sets for the \mathbb{C}^\times -action. Therefore, since ψ_n is \mathbb{C}^\times -equivariant, it suffices to show that $\psi_n(x_\lambda) = \mathbf{X}_\lambda$. This is precisely the statement of Theorem 5.4.8, which completes the proof of Theorem 1.2.1.

Let $N = n^2 - 1$. The Wronskian, definition 6.4.1, may be considered as a map $\text{Wr} : \text{Gr}_n(\mathbb{C}[x]_{2n}) \rightarrow \mathbb{P}^N$, where

$$\text{Wr}(W) = [c_0 : \dots : c_N] \quad \text{if} \quad \text{Wr}_W(x) = c_N x^N + \dots + c_1 x + c_0.$$

If $q = (q_1, \dots, q_n) \in \mathfrak{h}$, then its image in $\mathfrak{h}/\mathfrak{S}_n$ is $\mathbf{a} = (a_1, \dots, a_n)$ where

$$\prod_{i=1}^n (x - q_i) = x^n + a_n x^{n-1} + \dots + a_1.$$

We embed $\mathfrak{h}/\mathfrak{S}_n$ into \mathbb{P}^N , as a locally closed subvariety by

$$(a_1, a_2, \dots, a_n) \mapsto [a_1 : a_2 : \dots : a_n : 1 : 0 : \dots : 0]. \quad (7.1.1)$$

Proposition 7.1.2. *The map $\nu_n : X_n \rightarrow \mathcal{QE}_n$ restricts to an isomorphism of schemes*

$$\Omega_{0,\lambda,\mathbf{a}} \simeq \text{Wr}^{-1}(-\mathbf{a}) \cap \Omega_\lambda^{\text{qe}},$$

where the right hand side is the scheme theoretic intersection in $\text{Gr}_n(\mathbb{C}[x]_{2n})$.

Proof. By Theorem 1.2.1, ν_n restricts to an isomorphism of algebraic varieties $\Omega_\lambda \simeq \Omega_\lambda^{\text{qe}}$. By remark 5.2.2, it suffices to consider CM_n . As a locally closed embedding, $\beta_n : \Omega_\lambda^{\text{cm}} \hookrightarrow \text{Gr}_0(\mathbb{C}[x]_{2n})$ was given by the polynomial coefficients of the τ -function. Therefore, it suffices to show that

$$\begin{array}{ccc} \Omega_\lambda^{\text{cm}} & \xrightarrow{\beta_n} & \Omega_\lambda^{\text{qe}} \\ \varpi \downarrow & & \downarrow \text{Wr} \\ \mathfrak{h}/\mathfrak{S}_n & \xrightarrow{(-1)} & \mathfrak{h}/\mathfrak{S}_n \end{array} \quad (7.1.3)$$

commutes, as morphisms of schemes. For all (X, Y) in CM_n , we have $\det(X + \sum_{i=1}^\infty it_i(-Y)^{i-1}) = \tau_{\beta_n(X, Y)}(t_1, \dots)$, see (6.6.1). Setting $t_2 = t_3 = \dots = 0$ gives $\det(X + t_1) = \tau_{\beta_n(X, Y)}(t_1, 0, \dots)$. Equation (6.6.3) says that $\text{Wr}_{\beta_n(X, Y)}(t_1) = \tau_{\beta_n(X, Y)}(t_1, 0, \dots)$. Thus, $\det(X + t_1) = \text{Wr}_{\beta_n(X, Y)}(t_1)$. Since $\varpi(X, Y)$ is defined to be the coefficients of the polynomial $\det(t_1 - X)$, the diagram 7.1.3 commutes. \square

Remark 7.1.4. We have defined the Wronskian for any homogeneous space of quasi-exponentials. The proof of Proposition 7.1.2 shows that, as sets, we have

$$\nu_n(\Omega_{b, \lambda, a}) = \text{Wr}^{-1}(-a) \cap \Omega_{b, \lambda}^{\text{qe}} =: \Omega_{b, \lambda, -a}^{\text{qe}}$$

for all $a \in \mathfrak{h}/\mathfrak{S}_n$, $b \in \mathfrak{h}^*/\mathfrak{S}_n$, and λ a multipartition of type b .

7.2. Exponents. In this subsection, we will consider all spaces to be sets. Recall that, in addition to the generalized Verma modules, we also defined in (2.3.1) dual generalized Verma modules $\nabla(q, \mu)$. Considered as Z_n -modules, their supports were denoted $\mathcal{U}_{a, \mu}$, where $\bar{q} = a$ in $\mathfrak{h}/\mathfrak{S}_n$. In this section, we describe the sets $\nu_n(\mathcal{U}_{a, \mu})$.

Definition 7.2.1. Let $C \in \mathcal{QGr}$ be an n -dimensional space of quasi-exponentials. Then, the *sequence of exponents* of C at a point $b \in \mathbb{C} \cup \{\infty\}$ is the (unique) set of integers $\mathbf{d} = \{d_0 < \dots < d_{n-1}\}$ with the property that, for each i , there exists a function $f \in C$ with order d_i at b . A point b of $\mathbb{C} \cup \{\infty\}$ is said to be *singular* if the exponents of C at b differs from $\{0, \dots, n-1\}$.

Let $a = \sum_{i=1}^k n_i a_i \in \mathfrak{h}/\mathfrak{S}_n$, where the a_i are pairwise distinct. Choose a multipartition $\mu = (\mu^{(1)}, \dots, \mu^{(k)})$ of n such that $\mu^{(i)} \vdash n_i$. From μ we define the tuple of integers $\mathbf{d} = \{d_{i,j} \mid i = 1, \dots, k, j = 0, \dots, n_i - 1\}$ by

$$d_{i,j} := \mu_{n_i-j}^{(i)} + n_i - (j+1). \quad (7.2.2)$$

Then, set of all $C \in \mathcal{QE}$ such that the singularities of C are $\{a_1, \dots, a_k\}$ and the exponents of C at a_i are

$$\{0 < \dots < 2n - n_i - 1 < 2n - n_i + d_{i,1} < \dots < 2n - n_i + d_{i,n_i}\},$$

is denoted $\mathcal{U}_{a, \mu}^{\text{qe}}$. The parameterization is chosen so that we can make use of the following crucial result.

Proposition 7.2.3 ([30], Theorem 2.6). *For a, μ as above, we have $(\Omega_{a, \mu}^{\text{qe}})^B = \mathcal{U}_{a, \mu}^{\text{qe}}$ as subsets of \mathcal{QE} .*

As a consequence,

Theorem 7.2.4. *For all $q \in \mathfrak{h}$ with $a = \bar{q} \in \mathfrak{h}/\mathfrak{S}_n$, $\mu \in W_q$ and $b \in \mathfrak{h}^*/\mathfrak{S}_n$, the map ν_n gives bijections*

$$\begin{array}{ccc} \mathcal{U}_{a, \mu} & \xrightarrow{\sim} & \mathcal{U}_{a, \mu}^{\text{qe}} \\ \uparrow & & \uparrow \\ \mathcal{U}_{a, \mu, b} & \xrightarrow{\sim} & \mathcal{U}_{a, \mu, -b}^{\text{qe}} \end{array}$$

Proof. By remark 7.1.4, $\nu_n(\Omega_{b,\lambda,a}) = \Omega_{b,\lambda^t,-a}^{\text{qe}}$. As noted in section 5.5, the map ν_n intertwines the bispectral involution on X_n with Wilson's bispectral involution on \mathcal{QE} (or rather the corresponding integral transform as defined in [30]). Proposition 7.2.3 implies that $(\Omega_{a,\mu,b}^{\text{qe}})^B = \mathcal{U}_{a,\mu,b}^{\text{qe}}$.

Let $(-1) : H_n \xrightarrow{\sim} H_n$ be the isomorphism which is the identity on \mathfrak{S}_n and maps x_i to $-x_i$ and y_j to $-y_j$. Then, $B = F \circ (-)^* \circ (-1)$. We have $\Omega_{a,\mu,b}^{(-1)} = \Omega_{-a,\mu,-b}$. Proposition 5.7.2 implies that $\Omega_{-a,\mu,-b}^* = \Omega_{-a,\mu^t,b}$ and Lemma 5.6.1 (1) implies that $\Omega_{-a,\mu^t,b}^F = \mathcal{U}_{a,\mu^t,b}$. Therefore, $\Omega_{a,\mu,b}^B = \mathcal{U}_{a,\mu^t,b}$. This implies the claim of the theorem. \square

If $g(x)$ is a polynomial and $p \neq 0$, then the function $e^{px}g(x)$ has an irregular singularity of order one at infinity. Thus, if D is an n th order differential equation whose space of solutions is $C \in \mathcal{QGr}$ then D has only regular singularities in \mathbb{C} and (at worst) an irregular singularity at ∞ of order one. Moreover, the residue of D at ∞ is $\text{Supp}(C) \in \mathfrak{h}^*/\mathfrak{S}_n$. One should compare this with [3, Corollary 1] and section 6 of [28]. Recall that D is said to be *Fuchsian* if it has only regular singularities i.e. if and only if $\text{Supp}(C) = 0$.

Corollary 7.2.5. *The simple H_n -module L is Fuchsian if and only if the corresponding differential equation D_L is Fuchsian.*

Proof. The space C of solutions of D_L is a homogeneous space of quasi-exponential functions. As noted above, D_L will be Fuchsian if and only if the support of C equals zero. That is, if and only if the augmentation ideal in $\mathbb{C}[\mathfrak{h}^*]^{\mathfrak{S}_n}$ annihilates L . \square

Example 7.2.6. For each partition λ of n we have the simple H_n -module $L(\lambda)$. Since the support of $L(\lambda)$ is sent to the \mathbb{C}^\times -fixed point in $\Omega_\lambda^{\text{qe}}$, the proof of Lemma 6.5.1 shows that

$$D_{L(\lambda)} = \prod_{i=0}^{n-1} (x\partial - e_i),$$

where $e_i = n + \lambda_i - (i + 1)$.

Remark 7.2.7. We have associated to each simple H_n -module a differential equation D_L . We see that the properties of D_L such as its singularities and exponents translate into properties of the support of L .

7.3. Intersection of Schubert cells. In this subsection we assume that W is an arbitrary complex reflection group.

Conjecture 7.3.1. Choose $p \in \mathfrak{h}^*$, $q \in \mathfrak{h}$, $\lambda \in \text{Irr}(W_p)$ and $\mu \in \text{Irr}(W_q)$. Let $a \in \mathfrak{h}/W$ be the image of q and $b \in \mathfrak{h}^*/W$ the image of p . Assume that the support of both $\Delta(p, \lambda, a)$ and $\nabla(q, \mu, b)$ are contained in $X_c(W)_{\text{sm}}$. Let I denote the annihilator of the $Z_c(W)$ -module

$$\text{Hom}_{H_c(W)}(\nabla(q, \mu, b), \Delta(p, \lambda, a))$$

and set $Z_c(p, q, \lambda, \mu) = Z_c(W)/I$. Then, we conjecture that $Z_c(p, q, \lambda, \mu)$ is a Gorenstein ring and that the module $\text{Hom}_{H_c(W)}(\nabla(q, \mu, b), \Delta(p, \lambda, a))$ is isomorphic to the coregular (\simeq regular) representation as a $Z_c(p, q, \lambda, \mu)$ -module.

Now we return to the symmetric group. For $q_i \in \mathbb{C}$, let $\mathcal{F}(q_i)$ be the complete flag

$$\mathcal{F}_\bullet(q_i) : \mathcal{F}_j(q_i) = (x - q_i)^{2n-j} \mathbb{C}[x]_j, \quad 0 \leq j \leq 2n,$$

in $\mathbb{C}[x]_{2n}$. Let $q = (q_1, \dots, q_1, q_2, \dots, q_2, q_3, \dots)$, where the q_i are pairwise distinct and q_i occurs n_i terms. Let $\mu = (\mu^{(1)}, \dots, \mu^{(k)})$ be a multipartition with $\mu^{(i)} \vdash n_i$; equivalently $\mu \in \text{Irr}(\mathfrak{S}_q)$. Then,

we define

$$\Omega_{\mu}(q) = \bigcap_{i=1}^k \Omega_{\mu^{(i)}}(\mathcal{F}(q_i)),$$

a scheme theoretic intersection of Schubert cells in $\mathrm{Gr}_n(\mathbb{C}[x]_{2n})$. Let $\mathrm{Gr}_n(\mathbb{C}[x]_{2n})_{\mathrm{can}}$ denote the intersection of $\mathrm{Gr}_n(\mathbb{C}[x]_{2n})$ with $\mathcal{Q}\mathcal{E}$ in $\mathcal{Q}\mathrm{Gr}$, considered as a reduced variety. Then, $\mathrm{Gr}_n(\mathbb{C}[x]_{2n})_{\mathrm{can}} = \bigsqcup_{\lambda \vdash n} \Omega_{\lambda}^{\mathrm{ge}}$. We define $\Omega_{\mu}(q)_{\mathrm{can}}$ to be the scheme-theoretic intersection of $\Omega_{\mu}(q)$ with $\mathrm{Gr}_n(\mathbb{C}[x]_{2n})_{\mathrm{can}}$. In order to prove a special case of the above conjecture when $W = \mathfrak{S}_n$, we need to make the following technical assumption.

Assumption 7.3.2. We have an equality $\nu_n(\mathcal{U}_{\mathbf{a}, \mu, n, 0}) = \Omega_{\mu}(q)_{\mathrm{can}}$ as subschemes of $\mathrm{Gr}_n(\mathbb{C}[x]_{2n})$.

We remark that neither $\mathcal{U}_{\mathbf{a}, \mu, n, 0}$ or $\Omega_{\mu}(q)_{\mathrm{can}}$ is a reduced scheme. In order to convince the reader that assumption 7.3.2 is not unreasonable, we have

Lemma 7.3.3. *Let $q \in \mathfrak{h}$ and $\mu \in \mathrm{Irr}(\mathfrak{S}_q)$. Then, we have an equality $\nu_n(\mathcal{U}_{\mathbf{a}, \mu, n, 0}) = \Omega_{\mu}(q)_{\mathrm{can}}$ of subsets of $\mathrm{Gr}_n(\mathbb{C}[x]_{2n})$ and*

$$\dim \mathbb{C}[\mathcal{U}_{\mathbf{a}, \mu, n, 0}] = \dim \mathbb{C}[\Omega_{\mu}(q)_{\mathrm{can}}] = |\mathfrak{S}_n / \mathfrak{S}_q| \dim \mu.$$

Proof. A point $V \in \mathrm{Gr}_n(\mathbb{C}[x]_{2n})$ belongs to $\Omega_{\mu^{(i)}}(q_i)$ if and only if q_i is a singular point of V such that the exponents of V at q_i are encoded by $\mu^{(i)}$. On the other hand, Theorem 7.2.4 implies that $\nu_n(\mathcal{U}_{\mathbf{a}, \mu, n, 0})$ is the set of all *canonical* homogeneous spaces of quasi-exponentials with exponents prescribed by q and μ contained in $\mathrm{Gr}_n(\mathbb{C}[x]_{2n})$. Every space in $\mathrm{Gr}_n(\mathbb{C}[x]_{2n})$ is obvious homogeneous. Therefore $\nu_n(\mathcal{U}_{\mathbf{a}, \mu, n, 0})$ is the intersection of $\Omega_{\mu}(q)$ with $\mathrm{Gr}_n(\mathbb{C}[x]_{2n})_{\mathrm{can}}$, which by definition is $\Omega_{\mu}(q)_{\mathrm{can}}$.

To see that $\dim \mathbb{C}[\mathcal{U}_{\mathbf{a}, \mu, n, 0}] = |\mathfrak{S}_n / \mathfrak{S}_q| \dim \mu$, Theorem 3.3.9 implies that it suffices to notice that $e\nabla(q, \mu)$ is a free $\mathbb{C}[\mathfrak{h}^*]^{\mathfrak{S}_n}$ -module of rank $|\mathfrak{S}_n / \mathfrak{S}_q| \dim \mu$. This follows from the fact that, as a $\mathbb{C}[\mathfrak{h}^*]^{\mathfrak{S}_n}$ -module,

$$e\nabla(q, \mu) \simeq e(\mathbb{C}[\mathfrak{h}^*] \otimes \mathrm{Ind}_{\mathfrak{S}_q}^{\mathfrak{S}_n} \mu) \simeq e(\mathrm{Ind}_{\mathfrak{S}_q}^{\mathfrak{S}_n} (\mathbb{C}[\mathfrak{h}^*] \otimes \mu)) \simeq e_q(\mathbb{C}[\mathfrak{h}^*] \otimes \mu),$$

where e_q is the trivial idempotent in $\mathbb{C}\mathfrak{S}_q$.

Recall that two complete flags \mathcal{F}_{\bullet} and \mathcal{G}_{\bullet} in $\mathbb{C}[x]_{2n}$ are said to be transverse if $\dim \mathcal{F}_i \cap \mathcal{G}_j = \min\{i + j - 2n, 0\}$ for all i, j . Let $b \neq c \in \mathbb{C} \cup \{\infty\}$. Then, it is easy to check that the flags $\mathcal{F}_{\bullet}(b)$ and $\mathcal{F}_{\bullet}(c)$ are transverse. Hence, the flags appearing in the intersection $\Omega_{\mu}(q)$ are pairwise transverse. They are also transverse to $\Omega_{\bar{\lambda}}(\mathcal{F}(\infty))$ for each partition λ of n . As noted above, $\mathrm{Gr}_n(\mathbb{C}[x]_{2n})_{\mathrm{can}} = \bigsqcup_{\lambda \vdash n} \Omega_{\bar{\lambda}}(\mathcal{F}(\infty))$. Since the set-theoretic intersection $\Omega_{\mu}(q) \cap \Omega_{\bar{\lambda}}(\mathcal{F}(\infty))$ consists of finitely many points (notice that $\dim \Omega_{\mu^{(i)}}(\mathcal{F}(q_i)) = n^2 - |\mu^{(i)}|$ and $\dim \Omega_{\bar{\lambda}}(\mathcal{F}(\infty)) = |\lambda|$, hence if $\sum_i \dim \mu^{(i)} = n$ and $\lambda \vdash n$, then $\dim \Omega_{\mu}(q) \cap \Omega_{\bar{\lambda}}(\mathcal{F}(\infty)) = 0$) the transeverality condition implies that

$$[\Omega_{\mu^{(1)}}(\mathcal{F}(q_1))] \cdots [\Omega_{\mu^{(k)}}(\mathcal{F}(q_k))] \cdot [\Omega_{\bar{\lambda}}(\mathcal{F}(\infty))]$$

is some multiple of the identity in the cohomology ring $H^*(\mathrm{Gr}_0(\mathbb{C}[x]_{2n}))$, where $[X] \cdot [Y]$ denotes multiplication in $H^*(\mathrm{Gr}_0(\mathbb{C}[x]_{2n}))$ of the classes defined by the *closures* of the locally closed subvarieties X, Y of $\mathrm{Gr}_0(\mathbb{C}[x]_{2n})$. Thus,

$$\dim \mathbb{C}[\Omega_{\mu}(q)_{\mathrm{can}}] = \sum_{\lambda \vdash n} [\Omega_{\mu^{(1)}}(\mathcal{F}(q_1))] \cdots [\Omega_{\mu^{(k)}}(\mathcal{F}(q_k))] \cdot [\Omega_{\bar{\lambda}}(\mathcal{F}(\infty))].$$

Let $\sigma_{\lambda} = [\Omega_{\lambda}(\mathcal{F}(b))] = [\Omega_{\lambda}(\mathcal{F}(\infty))]$ be the class of a Schubert cell in $H^*(\mathrm{Gr}_0(\mathbb{C}[x]_{2n}))$. They form a basis of $H^*(\mathrm{Gr}_0(\mathbb{C}[x]_{2n}))$ such that $\sigma_{(n, \dots, n)} = 1$. Let $\langle -, - \rangle$ be the non-degenerate pairing on $H^*(\mathrm{Gr}_0(\mathbb{C}[x]_{2n}))$ defined by letting $\langle [X], [Y] \rangle$ be the coefficient of 1 in the expansion of $[X] \cdot [Y]$

in terms of the basis $\{\sigma_\lambda\}$. The duality theorem, [15, page 149], says that $\langle \sigma_\lambda, \sigma_{\bar{\mu}} \rangle = \delta_{\lambda, \mu}$. Thus, Schubert calculus implies that

$$\begin{aligned} \dim \mathbb{C}[\Omega_{\mu}(q)_{\text{can}}] &= \sum_{\lambda \vdash n} \langle \sigma_{\mu^{(1)}} \cdots \sigma_{\mu^{(k)}}, \sigma_\lambda \rangle \\ &= \sum_{\lambda \in \text{Irr}(\mathfrak{S}_n)} \text{Hom}_{\mathbb{C}\mathfrak{S}_n}(\lambda, \text{Ind}_{\mathfrak{S}_q}^{\mathfrak{S}_n} \mu) = \dim \text{Ind}_{\mathfrak{S}_q}^{\mathfrak{S}_n} \mu, \end{aligned}$$

as required. \square

Theorem 7.3.4. *Under assumption 7.3.2, ν_n induces an isomorphism of Gorenstein rings*

$$Z_{\mathbf{c}}(0, q, \lambda, \mu) \simeq \mathbb{C}[\Omega_{0, \lambda, -\mathbf{a}}^{\text{qe}} \cap \Omega_{\mu}(q)],$$

and $\text{Hom}_{H_n}(\nabla(q, \mu, 0), \Delta(0, \lambda, \mathbf{a}))$ is the coregular representation as a $Z_{\mathbf{c}}(0, q, \lambda, \mu)$ -module.

Proof. The Morita equivalence between H_n and Z_n implies that

$$\text{Hom}_{H_n}(\nabla(q, \mu, 0), \Delta(0, \lambda, \mathbf{a})) \simeq \text{Hom}_{Z_n}(e\nabla(q, \mu, 0), e\Delta(0, \lambda, \mathbf{a})).$$

Let I be the annihilator of $e\nabla(q, \mu, 0)$ in Z_n and J the annihilator of $e\Delta(0, \lambda, \mathbf{a})$. Then, we have shown that $e\nabla(q, \mu, 0) \simeq Z_n/I$ and $e\Delta(0, \lambda, \mathbf{a}) \simeq Z_n/J$ are cyclic Z_n -modules. By Theorem 3.3.9 and Proposition 7.1.2, we have

$$\text{Spec } Z_n/J = \Omega_{0, \lambda, \mathbf{a}} \simeq \text{Wr}^{-1}(-\mathbf{a}) \cap \Omega_{\lambda}^{\text{qe}} = \Omega_{0, \lambda, -\mathbf{a}}^{\text{qe}}.$$

Using assumption 7.3.2, we have $\text{Spec } Z_n/I = \mathcal{U}_{\mathbf{a}, \mu, n, 0} \simeq \Omega_{\mu}(q)_{\text{can}}$. Therefore, [28, Lemma 4.3] implies that

$$Z_n/(I + J) \simeq \mathbb{C}[\text{Wr}^{-1}(-\mathbf{a}) \cap \Omega_{\lambda}^{\text{qe}} \cap \Omega_{\mu}(q)_{\text{can}}] = \mathbb{C}[\text{Wr}^{-1}(-\mathbf{a}) \cap \Omega_{\lambda}^{\text{qe}} \cap \Omega_{\mu}(q)]$$

is a Gorenstein ring. This proves the first statement of the theorem. The result [28, Lemma 3.8] states that:

Claim 7.3.5. Let Z be a commutative ring and I, J ideals of Z such that

- $\dim Z/I, Z/J < \infty$,
- Z/J and $Z/I + J$ are Gorenstein.

Let $\bar{I} = I + J$ in Z/J . Then, $\ker \bar{I} \simeq (Z/(I + J))^*$ as $Z/(I + J)$ -modules.

Applying the above claim in our case, it suffices to identify $\ker \bar{I}$ with

$$\text{Hom}_{H_n}(\nabla(q, \mu, 0), \Delta(0, \lambda, \mathbf{a})) = \text{Hom}_{Z_n}(Z_n/J, Z_n/I).$$

This is straight-forward. \square

Remark 7.3.6. The claim about dimensions made after Corollary 1.3.1 can be deduced from the proof of Lemmata 2.3.3 and 7.3.3. Also, $\text{Hom}_{H_n}(\nabla(q, \mu, 0), \Delta(0, \lambda, \mathbf{a})) = \text{Hom}_{H_n}(\nabla(q, \mu), \Delta(0, \lambda, \mathbf{a}))$, which implies that Corollary 1.3.1 is equivalent to the statement of Theorem 7.3.4.

8. APPENDIX: BATALIN-VILKOVISKI STRUCTURES

In this appendix we summarize the main results of [1], as required in this article. We follow the presentation of *loc. cit.*, which the reader is encouraged to consult for further details. All undecorated tensor products will mean tensor product over \mathbb{C} .

Let $D = \bigoplus_{i \geq 0} D_i$ be a graded commutative algebra. If D is equipped with a map $\delta : D_\bullet \rightarrow D_{\bullet-1}$ such that $\delta^2 = 0$ and⁴

$$\begin{aligned} \delta(abc) &= \delta(ab)c + (-1)^{\deg(a)} a\delta(bc) + (-1)^{(\deg(a)+1)\deg(b)} b\delta(ac) - \delta(a)bc \\ &\quad - (-1)^{\deg(a)} a\delta(b)c - (-1)^{\deg(a)+\deg(b)} ab\delta(c) + \delta(1)abc, \end{aligned}$$

for all homogeneous elements a, b, c of D , then D is said to be a *Batalin-Vilkoviski* (BV) algebra. Every Batalin-Vilkoviski algebra has in addition the structure of a *Gerstenhaber* algebra. Namely, for x, y homogeneous elements in D , the formula

$$[x, y] := \delta(xy) - \delta(x)y - (-1)^{\deg x} x\delta(y)$$

makes D_\bullet into a Gerstenhaber algebra. This means that $[-, -]$ has degree -1 and

$$\begin{aligned} [a, bc] &= [a, b]c + (-1)^{(\deg(a)-1)\deg(b)} b[a, c], \\ [a, b] &= -(-1)^{(\deg(a)-1)(\deg(b)-1)} [b, a], \\ [a, [b, c]] &= [[a, b], c] + (-1)^{(\deg(a)-1)(\deg(b)-1)} [b, [a, c]], \end{aligned}$$

for all homogeneous elements a, b, c of D i.e. the graded commutative algebra D is an odd Poisson algebra.

Let X be a smooth affine variety equipped with a symplectic two-form ω . Let Y be a smooth, coisotropic subvariety of X . Let $\mathbf{N}_{X/Y}$ denote the sheaf of sections of the normal bundle of Y in X . It is a sheaf of \mathcal{O}_X -modules supported on Y . Its dual $\mathbf{N}_{X/Y}^\vee$ is the sheaf of sections of the conormal bundle. Since the two-form ω is non-degenerate, it induces an isomorphism $\Gamma(X, \Omega_X^2) \simeq \Gamma(X, \wedge^2 \Theta_X)$. We let P be the image of ω under this isomorphism; it is a Poisson bivector. The graded commutative algebras $\wedge^\bullet \mathbf{N}_{X/Y}^\vee$ and $\wedge^\bullet \mathbf{N}_{X/Y}$ have a natural structure of BV algebra. Namely, the differential on $\wedge^\bullet \mathbf{N}_{X/Y}^\vee$ is given by the formula $\delta = i_P \circ d_{\text{DR}} + d_{\text{DR}} \circ i_P$, where d_{DR} is the deRham differential of degree one and i_P denotes contraction with P (and thus δ has degree -1). The differential on $\wedge^\bullet \mathbf{N}_{X/Y}$ is given by the Schouten bracket $[P, -]$. That these operations are indeed well-defined follows from the fact that the ideal defining Y is involutive. Combining Corollary 1.1.3, Propositions 5.1.1 and 5.2.1 of [1] gives

Theorem 8.0.7. *The sheaf of graded algebras $\mathcal{T}or_\bullet^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)$ admits a canonical structure of a Gerstenhaber algebra such that*

$$\mathcal{T}or_\bullet^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \wedge^\bullet \mathbf{N}_{X/Y}^\vee \quad (8.0.8)$$

as Gerstenhaber algebras.

Moreover, the sheaf $\mathcal{E}xt_{\mathcal{O}_X}^\bullet(\mathcal{O}_Y, \mathcal{O}_Y)$ admits a canonical structure of Gerstenhaber module over the Gerstenhaber algebra $\mathcal{T}or_\bullet^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)$ such that

$$\mathcal{E}xt_{\mathcal{O}_X}^\bullet(\mathcal{O}_Y, \mathcal{O}_Y) \simeq \wedge^\bullet \mathbf{N}_{X/Y} \quad (8.0.9)$$

as Gerstenhaber modules, compatible in the obvious sense with the identification (8.0.8).

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⁴This equation is equivalent to the fact that δ is a differential operator of order at most two.

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